Problems with lack of compactness

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Program

Compactness is a crucial ingredient in variational problems since it guarantees, for instance, that minimizing or Palais-Smale sequences admit convergent subsequeces in the function space where the problem is set. In this way, compactness allows to find critical points of the energy functional associated to the problem. A typical manifestation of the lack of compactness is that such sequences are bounded but not pre-compact. Lack of compactness can be caused by different reasons and may require different tools to be overcome.

In this course, we will look for H^1-solutions to some semilinear elliptic problems of the form $-\Delta u + q(x)u = |u|^{p-2}u$, where compactness may fail due to the unboundedness of the domain or the (super)critical growth of the nonlinearity. More precisely, concerning *the problem in* $\mathbb{R}^N$ *with subcritical nonlinearity*,

- for $q \equiv 1$, we will find a radial solution exploiting the symmetries of the problem, via the Principle of Symmetric Criticality by Palais;
- for q unbounded and non-radial, we will find a solution working in a smaller space where compactness is restored;
- for q bounded and non-radial, we will find a solution via the Concentration-Compactness Principle by Lions.

We will then consider the case with *the nonlinearity having critical or supercritial growth*. In particular,

- we will prove the existence of extremal functions for the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, and prove non-existence in bounded domains;
- in bounded domains, we will study existence for the Brezis-Nirenberg problem;
- in a ball, under Neumann boundary conditions, we will find a radial solution to the supercritical Lin-Ni-Takagi problem.

References

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