Titolo del corso. Tropical and algebraic geometry of curves

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Membro del collegio proponente. ???

Ore frontali di lezione. 15+15 hours

Periodo di lezione. Nov 2024 - Apr 2025

Settore/i disciplinare del corso. Algebra e geometria

Tipologia di corso: Base

Modalità di verifica dell'apprendimento. Esame orale/seminario

Abstract del corso. In the first part of the course, we introduce tropical geometry with an emphasis on tropical curves. Although they are nothing but metric graphs, there is a very rich and well-developed theory entirely parallel to that of Riemann surfaces. We will focus on counts of tropical plane curves, and on the divisor theory of metric graphs.

In the second part of the course, we study the algebro-geometric counterpart of the objects presented in the first part. We introduce the moduli space of curves and describe some of its properties. Finally, we focus on moduli of linear series with enumerative applications.

Programma del corso. We introduce tropical geometry with an emphasis on the theory for the curve case, with a view towards sketching two important applications of the theory: enumerative geometry for counting curves satisying suitable point conditions, and Brill-Noether theory for proving non-existence of certain divisors on general algebraic curves. The first example is an application of embedded tropical curves, which takes us to the notion of tropical vanishing, embedded tropical varieties and the fundamental theorem of tropical geometry for the relation between classical algebraic curves and tropical curves. The heart of this application is Mikhalkin's correspondence theorem and the resolution of a combinatorial problem using *diagrammes en étage*, which counts certain family of graphs. The second application leads us to introduce abstract algebraic curves as metric graphs, a divisor theory on them with linear equivalence and rational functions, and the study of a particular class of graphs called chains of loops. Here the correspondence theorem is given by Baker's specialization lemma, and the combinatorial problem reduces to a count of Young Diagrams, same that come up in representaiton theory. Time permiting we will also initiate the study of the higher dimensional setting, whose building blocks is given by Bergman Fans associated to matroids, and whose linear varieties are cryptomorphic to valuated matroids.

The main bibliography for the fundamentals of embedded tropical curves is [BIMS15; MS15], and the topics comprise:

- Tropical algebra $(\mathbb{R} \cup \{\infty\}, \min, +)$
- Tropical varieties
- Drawing tropical varieties and Newton Polygon
- Fundamental theorem of tropical geometry

The bibliography for enumerative geometry is [Bru08; Mik05] and the topics comprise:

- The counting problem and its space of parameters
- Tropical correspondence theorem
- Points in general position
- Properties of Diagrammes en étage
- Counting diagrammes en étage

The bibliography for abstract tropical curves, divisor theory and tropical brill noether theory comprise [BN07; Bak08; CDPR12].

- Metric graphs and divisor theory
- Baker's specialization lemma
- Chains of loops
- Counting Young diagrams

In the second part of the course, we introduce moduli theory with a focus on the moduli space of curves and its compactification due to Deligne and Mumford (see for instance [Alp24]).

Depending on the participants' interest and background, we will then cover one or more of the following applications:

- spaces of linear series and basics of Brill–Noether theory [ACGH13],
- admissible covers/limit linear series and the Kodaira dimension of the moduli space of curves [EH87],
- stable maps (Gromov–Witten theory) and Kontsevich's formula for the number of rational plane curves [FP97],
- basics of logarithmic geometry with a view towards tropicalising the moduli space of curves [ACP15] and other correspondence theorems.

References

[ACGH13] Enrico Arbarello et al. *Geometry of Algebraic Curves: Volume I*. Vol. 267. Springer Science & Business Media, 2013. ISBN: 1-4757-5323-3.

[ACP15] Dan Abramovich, Lucia Caporaso, and Sam Payne. "The Tropicalization of the Moduli Space of Curves". In: *Ann. Sci. Éc. Norm. Supér.(4)* 48.4 (2015), pp. 765–809. ISSN: 0012-9593. arXiv: 1212.0373v2.

[Alp24] Jarod Alper. *Stacks and moduli*. 2024.

[Bak08] Matthew Baker. "Specialization of Linear Systems from Curves to Graphs". In: *Algebra & Number Theory* 2.6 (2008), pp. 613–653. ISSN: 1944-7833. arXiv: 0701075v4.

[BIMS15] Erwan Brugallé et al. "Brief Introduction to Tropical Geometry". In: Proceedings of the Gökova Geometry-Topology Conference. 2015, pp. 1–75.

[BN07] Matthew Baker and Serguei Norine. "Riemann–Roch and Abel–Jacobi Theory on a Finite Graph". In: *Advances in Mathematics* 215.2 (2007), pp. 766–788. ISSN: 0001-8708. arXiv: 0608360v3.

[Bru08] Erwan Brugallé. "Géométries Énumératives Complexe, Réelle et Tropicale". In: *Géométrie tropicale* (2008), pp. 27–84.

[CDPR12] Filip Cools et al. "A Tropical Proof of the Brill–Noether Theorem". In: *Advances in Mathematics* 230.2 (2012), pp. 759–776. ISSN: 0001-8708.

[EH87] David Eisenbud and Joe Harris. "The Kodaira dimension of the moduli space of curves of genus $\geq$ 23". In: *Invent. Math.* 90.2 (1987), pp. 359–387. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01388710. URL: https://doi.org/10.1007/BF01388710.

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[Mik05] Grigory Mikhalkin. "Enumerative Tropical Algebraic Geometry in $\mathbb Rˆ2$". In: *Journal of the American Mathematical Society* 18.2 (2005), pp. 313–377. ISSN: 0894-0347. arXiv: 0312530.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to Tropical Geometry*. Vol. 161. American Mathematical Soc., 2015. ISBN: 0-8218-5198-5.