Lectures in holomorphic function theory

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Preface

These are the notes for an "entry level" Graduate Course in Holomorphic Function Theory at the University of Bologna (2025). Although the attendees surely had previous exposure to holomorphic functions (the equivalence of various definitions of holomorphicity, local behavior, singularities, residues...), the basics of the theory are reviewed with proofs in the first few lectures. Some notions from Advanced Calculus, Lebesgue integration, Topology, and Functional Analysis will be freely used during the course (but the most crucial ones are proved in an appendix). The prerequisites can be found, e.g., in some chapters of [**Rudin**] (Advanced Calculus), [**Bass**] (Lebesgue integration, Measure Theory, Functional Analysis), [**Spanier**] (Topology). I am indebted with several books and lecture notes on Holomorphic Theory. Let me mention [**Ahlfors**] (where I learned the basics, the instructor being Albert Baernstein II, soon after my advisor), [**Andersson**], [**Nehari**], [**Sarason**], and [**Tao**].

The plan of the course is covering some basic results which are not in the intersection of the usual introductory courses on holomorphic theory (e.g. the Riemann mapping theorem), then moving to a narrow selection of some more advanced topics, such as infinite products and the relation between zeros and growth of a function. It would be nice to have the time to go through the main facts of H^p theory, which has been a central topic of interest since the mid XX century, also in view of its deep connections with Operator Theory and Holomorphic Control Theory.

These notes are a version of the "theoretical minimum" of holomorphic function theory. For advanced topics covered in the class I will produce *ad hoc* notes.

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CHAPTER 1

Foundations

This chapter presents, together with the one on harmoc functions, a biased version of the *theoretical minimum* of holomorphic theory, which is typically covered in an introductory one-semester course. There is consensus that the minimum consists in proving that holomorphic functions can be defined in several equivalent ways, and that some basic phenomena shedding light on the nature of holomorphicity follow from this with not too much effort. Some of these phenomena have a more geometric nature, and others have a more analytic nature. The selection of the phenomena, and the way they are presented, is debatable. I have made here my choices, sometimes following old and new texts, sometimes following my own taste. I have made an effort to present the material in a natural and intuitive way; but of course this has to be judged by the students.

There are some departures from most texts on the subjects. One is that Green's theorem is used from the start. This shortens several proofs, it gives a better understanding of the Cauchy kernel as the "fundamental solution" of the $\bar{\partial}$ equation (which will be used in some more advanced chapters), and it helps to highlight similarities and differences between the theory of holomorphic functions and that of harmonic functions. After all, Green's theorem and the Cauchy integral theorem lead to the same notion of "cancellation" in the boundary terms. I got this idea from [**Andersson**], although I assume less prerequisites from real analysis than he does. The second is that the four basic characterizations of holomorphic functions are concentrated in the first few lectures: existence of the complex derivative, Cauchy-Riemann equations, local Cauchy theorem (or Morera theorem), expansion in power series. Once the equivalence of different viewpoints is grasped, I believe, manipulating concepts, pictures and calculations becomes much easier. Third, the discussion of subtle topological issues is postponed to after the Riemann mapping theorem. As observed in [**Ahlfors**] p.231, the Riemann mapping theorem has a strong topological content. Following Koebe's proof, it implies that if the square root of a holomorphic function can be defined on an open set Ω which is not the whole plane, then the set can be bi-holomorphically (hence, homeomorphically) mapped onto a disc. This endows Ω with a privileged system of coordinates, using which homotopy can be easily handled.

In this chapter there is no use of Lebesgue theory or of Functional Analysis, with the exception of the Ascoli-Arzelà theorem, which routinely taught in Advanced Calculus, or in Topology.

1. Calculus in complex coordinates

Here we translate in complex coordinates some foundational results from Advanced Calculus. Several basic results about holomorphic functions, as we shall see, are an immediate consequence of real variable results. Sometimes this is not the best point of view: holomorphic calculus *per se* has its *raison d'être*. But in some cases it is necessary, or faster, to pass through real variables, and some geometric phenomena are best seen as facts concerning functions of two real variables. The open disk centered at a and having radius r is denoted by $D(a,r)$.

The complex plane $\mathbb{C}$ is identified with $\mathbb{R}^2$, $z = x+iy \equiv (x,y)$, so that the modulus in $\mathbb{C}$ is identified with the Euclidean norm in $\mathbb{R}^2$: $|z| = \sqrt{x^2+y^2}$. This way, metric, topological and differential notions in $\mathbb{R}^2$ (openness, continuity, compactness...) are immediately translated into corresponding notions in $\mathbb{C}$. In most cases we leave to the reader making sense of what a metric, or topological, or differential notation means in the complex plane.

1.1. Derivatives.

1.1.1. *Complex partial derivatives.* We will often deal with the complex differentials $dz = dx + idy$ and $d\bar{z} = dx - idy$. For functions $f : \Omega \to \mathbb{C}$ defined on open $\Omega \subseteq \mathbb{C}$, we will use the *complex partial derivatives* (or *Wirtinger operators*, or *Cauchy-Riemann operators*):

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),\ \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \tag{1.1}$$

We will also use the notation

$$\partial f = \partial_z f = f_z,\ \bar{\partial} f = \partial_{\bar{z}} f = f_{\bar{z}}.$$

The reason to introduce the complex partial derivatives is that they are the coefficients in the 1^{st} order Taylor expansion of f with respect to $dz, d\bar{z}$ (assuming f is differentiable):

$$\begin{aligned}
df(z) &= \partial_x f(z)dx + \partial_y f(z)dy \\
&= \partial_x f(z)\left(\frac{dz+d\bar{z}}{2}\right) + \partial_y f(z)\left(\frac{dz-d\bar{z}}{2i}\right) \\
&= \frac{1}{2}\left(\partial_x f(z) - i\partial_y f(z)\right)dz + \frac{1}{2}\left(\partial_x f(z) + i\partial_y f(z)\right)d\bar{z} \\
&= \partial_z f(z)dz + \partial_{\bar{z}} f(z)d\bar{z}, \qquad (1.2)
\end{aligned}$$

where all products are in $\mathbb{C}$.

If $f : E \supseteq \mathbb{C} \to \mathbb{C}$, we write $\bar{f}(z) := \overline{f(z)}$. That might cause some confusion because in many texts and articles on holomorphic function theory the same notation is used differently: $\bar{f}(z) := \overline{f(\bar{z})}$.

Exercise 1. (i) *Show that* $\overline{\partial_{\bar{z}} f} = \partial_z \bar{f}$, $\overline{\partial_z f} = \partial_{\bar{z}} \bar{f}$.
(ii) *Show the chain rule in complex coordinates,*

$$\partial_z(g \circ f) = \partial_w g \partial_z f + \partial_{\bar{w}} g \partial_z \bar{f},$$
$$\partial_{\bar{z}}(g \circ f) = \partial_w g \partial_{\bar{z}} f + \partial_{\bar{w}} g \partial_{\bar{z}} \bar{f}.$$

(iii) *Show that* $\Delta f = 4\partial_z\partial_{\bar{z}} f = 4\partial_{\bar{z}}\partial_z f$, *where* $\Delta = \partial_{xx} + \partial_{yy}$ *is the Laplace operator.*

Exercise 2. (i) *Let* $n \geq 0$ *be an integer. Show that* $\partial(z^n) = nz^{n-1}$ *and* $\bar{\partial}(z^n) = 0$. *On the other hand,* $\bar{\partial}(\bar{z}^n) = n\bar{z}^{n-1}$ *and* $\partial(\bar{z}^n) = 0$.
(ii) *Let* $f(z) = z^n = u(z) + iv(z)$. *Show that* $\Delta u = \Delta v = 0$.
(iii) *Recall that* $e^z := e^x(\cos(y) + i\sin(y))$. *Verify that* $\bar{\partial}(e^z) = 0$ *and* $\partial(e^z) = e^z$.

1.1.2. *Functions as maps.* Functions $f : \mathbb{C} \supseteq \Omega \to \mathbb{C}$ might be viewed as (i) transformations of the plane (or, dually, changes of coordinates); (ii) vector fields. Here we think of a differentiable f as a transformation and we see how it acts at the infinitesimal level.

The map $w = f(z)$ is *orientation preserving* at a if its Jacobian has positive determinant. Since

$$\begin{aligned} |\partial f(a)|^2 - |\bar{\partial} f(a)|^2 &= 1/4[(u_x + v_y)^2 + (v_x - u_y)^2 - (u_x - v_y)^2 - (v_x + u_y)^2] \\ &= u_x v_y - v_x u_y \\ &= \det Jf(a), \end{aligned}$$

we have that f is orientation preserving at a if and only if:

$$|\partial f(a)| > |\bar{\partial} f(a)|. \tag{1.3}$$

Assume f is orientation preserving at a. The differential $df(a)$ of f at a is a linear map on the plane, which is determined by its values on the unit circle, which is mapped to an ellipse $\mathcal{E}$. With $h = e^{it}$, we compute (I use $[\cdot]$ for the argument, not to create confusion with the complex product):

$$\begin{aligned} df(a)[h] &= \partial f(a) h + \bar{\partial} f(a) \bar{h} \\ &= \partial f(a) h \left(1 + \frac{\bar{\partial} f(a)}{\partial f(a)} \bar{h}^2\right) \\ &= \partial f(a) e^{it} \left(1 + \mu(a) e^{-2it}\right), \end{aligned}$$

where

$$\mu_f(a) = \frac{\bar{\partial} f(a)}{\partial f(a)} \tag{1.4}$$

is the *Beltrami coefficient* of f at a.

The ratio between the major and minor axis of the ellipse $\mathcal{E}$ is then

$$K_f(a) = \frac{1 + |\mu_f(a)|}{1 - |\mu_f(a)|}, \tag{1.5}$$

which is called the *dilatation* of f at a. The major and minor half-axis, in fact, have lengths

$$\begin{aligned} L &= |\partial f(a)| + |\bar{\partial} f(a)| = \\ l &= |\partial f(a)| - |\bar{\partial} f(a)|, \end{aligned}$$

which, if $\bar{\partial} f(a) \neq 0$, correspond to the directions

$$\begin{aligned} t(L) &= \frac{1}{2} \arg\left(\frac{\bar{\partial} f(a)}{\partial f(a)}\right) = \frac{1}{2} \arg(\mu_f(a)), \\ t(l) &= \frac{1}{2} \arg(\mu_f(a)) + \frac{\pi}{2} \end{aligned}$$

in the z-plane. In the w plane, the major/minor axis of $\mathcal{E}$ are in the directions given by

$$\begin{aligned} \arg\left(df(a)[e^{it(L)}]\right) &= \arg(\partial f(a)) - t(L) = \arg(\partial f(a)) - \frac{1}{2} \arg(\mu_f(a)), \\ \arg\left(df(a)[e^{it(l)}]\right) &= \arg(\partial f(a)) - t(L) + \frac{\pi}{2} = \arg(\partial f(a)) - \frac{1}{2} \arg(\mu_f(a)) + \frac{\pi}{2}. \end{aligned}$$

Passing from the linear approximation to the function f, and using the definition of differentiability, we have the following.

LEMMA 1. *Let $f : \Omega \to \mathbb{C}$ be differentiable at a and orientation preserving. Then,*

$$\limsup_{\epsilon\to 0}\{|f(z+h)-f(z)|/|h| : |h|\le\epsilon\} = |\partial f(a)| + |\bar\partial f(a)|$$

$$\liminf_{\epsilon\to 0}\{|f(z+h)-f(z)|/|h| : |h|\le\epsilon\} = |\partial f(a)| - |\bar\partial f(a)|$$

$$\lim_{\epsilon\to 0}\frac{\sup\{|f(z+h)-f(z)| : |h|\le\epsilon\}}{\inf\{|f(z+k)-f(z)| : |k|\le\epsilon\}} = \frac{|\partial f(a)|+|\bar\partial f(a)|}{|\partial f(a)|-|\bar\partial f(a)|}. \tag{1.6}$$

The first limit is achieved along $h = \epsilon e^{it}$, with

$$t = \frac{1}{2}\arg\left(\frac{\bar\partial f(a)}{\partial f(a)}\right); \tag{1.7}$$

while the second is achieved along $h = \epsilon e^{is}$, with

$$s = \frac{1}{2}\arg\left(\frac{\bar\partial f(a)}{\partial f(a)}\right) + \frac{\pi}{2} = t + \frac{\pi}{2}. \tag{1.8}$$

1.2. Line integrals and Green's theorem (Pompeiu's formula). If $c : [a,b] \to E$ is a piecewise smooth curve in a subset of and $f : E \to \mathbb{C}$ is continuous,

$$\int_c f(z)dz := \int_a^b f(c(t))\dot c(t)dt,$$

where the product inside the integral is the complex one. Let $f = u + iv$. Besides the usual properties of line integrals, we have

$$\left|\int_c f(z)dz\right| \le \int_c |f(z)|\cdot|dz|. \tag{1.9}$$

In fact, there is some real s such that

$$\begin{aligned}\left|\int_c f(z)dz\right| &= e^{is}\int_c f(z)dz = \int_a^b e^{is} f(c(t))\dot c(t)dt\\ &= \int_a^b \mathrm{Re}\left(e^{is}f(c(t))\dot c(t)\right)dt\\ &\le \int_a^b \left|e^{is}f(c(t))\dot c(t)\right|dt = \int_a^b |f(c(t))|\cdot|\dot c(t)|\,dt\\ &= \int_\gamma |f(z)|\cdot|dz|.\end{aligned}$$

It is easy to make the following calculation rigorous:

$$\begin{aligned}\int_c f(z)dz &= \int_c (u+iv)(dx+idy)\\ &= \int_c [(udx - vdy) + i(vdx+udy)]\\ &= \int_c (\alpha + i\beta),\end{aligned} \tag{1.10}$$

where

$$\alpha = udx - vdy \text{ and } \beta = vdx + udy \tag{1.11}$$

are 1-forms.

In the appendix we have two versions of Green's theorem: the more local theorem 62, and the more global theorem 63, which extends to domains with piecewise C^1 boundaries. In complex coordinates, Green's theorems have the following form.

THEOREM 1 (Green's theorem in complex coordinates). *Let Ω be a domain (open, connected) in $\mathbb{C}$, which satisfies the hypothesis of theorem 62 or theorem 63. If $f : cl(\Omega) \to \mathbb{C}$ is continuous on $cl(\Omega)$, and C^1 on Ω with integrable partial derivatives, then*

$$-\int_\Omega \bar\partial f(z) dz \wedge d\bar z = 2i \int_\Omega \bar\partial f(z) dx \wedge dy = \int_{\partial\Omega} f(z) dz. \tag{1.12}$$

We take here the wedge product of complex 1-forms,

$$\frac{i}{2} dz \wedge d\bar z = \frac{i}{2}(dx + idy) \wedge (dx - idy) = dx \wedge dy,$$

where the latter expression denotes the area element on the plane.

PROOF. We think of $f = u + iv$ as of the vector field (u, v).

$$\begin{aligned}
R.H.S.((1.12)) &= \int_c (u+iv)(dx+idy) \\
&= \int_c [(udx - vdy) + i(vdx + udy)] \\
&= \int_\Omega [(-v_x - u_y) + i(u_x - v_y)] dx \wedge dy \\
&= i\int_\Omega [i(v_x + u_y) + (u_x + i^2 v_y)] dx \wedge dy \\
&= i\int_\Omega [(u+iv)_x + i(u+iv)_y] dx \wedge dy \\
&= 2i \int_\Omega \bar\partial f dx \wedge dy.
\end{aligned}$$

□

One of the most useful, classical consequences of Green's theorem is that we can find the *Green function* of the Laplace operator. In complex coordinates, a slight variation of the proof shows that $\frac{1}{\pi z}$ is the *Green function* of $\bar\partial$. The key fact here is that

$$\bar\partial(1/z) = 0 \text{ for } z \neq 0, \tag{1.13}$$

and by translation $\bar\partial(1/(z-a)) = 0$ when $z \neq a$.

EXERCISE 3. *Show that $\bar\partial(1/z) = 0$ for $z \neq 0$.*

As a consequence, if f is C^1,

$$\bar\partial \left(\frac{f(z)}{z-a} \right) = \frac{\bar\partial f(z)}{z-a} \text{ if } z \neq a.$$

THEOREM 2 (Pompeiu's formula). *Let c and Ω be as in theorem 1, and let f be a C^1, complex valued function defined in $cl(\Omega)$. For a in Ω,*

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} dz - \frac{1}{\pi} \int_\Omega \frac{\bar\partial f(z)}{z-a} dxdy. \tag{1.14}$$

PROOF. We apply theorem 1 to $\Omega_\epsilon = \Omega \setminus \mathrm{cl}(D(a,\epsilon))$, with the function $\frac{f(z)}{z-a}$. Since $z \mapsto \frac{1}{z-a}$ satisfies Cauchy-Riemann equations in Ω_ϵ, $\partial_{\bar z}(f(z)/(z-a)) = (\partial_{\bar z} f(z))/(z-a)$. Let $c_\epsilon(t) = a + \epsilon e^{it}$. $|t| \le \pi$:

$$\begin{aligned}\frac{1}{\pi}\int_{\Omega_\epsilon}\frac{\partial_{\bar z} f(z)}{z-a}dxdy - \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz &= -\frac{1}{2\pi i}\int_{-\pi}^{+\pi}\frac{f(a+\epsilon e^{it})}{\epsilon e^{it}}\epsilon i e^{it}dt \\ &= -\frac{1}{2\pi}\int_{-\pi}^{+\pi} f(a+\epsilon e^{it})dt \\ &\to -f(a)\end{aligned}$$

as $\epsilon \to 0$. On the other hand, $\frac{\partial_{\bar z} f(z)}{z-a}$ is absolutely integrable in Ω, hence, by dominated convergence,

$$\frac{1}{\pi}\int_{\Omega}\frac{\partial_{\bar z} f(z)}{z-a}dxdy - \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz = -f(a),$$

as wished. □

EXERCISE 4. *Let n be an integer. Verify that*

$$\frac{1}{2\pi i}\int_{|z|=r} z^n dz = \begin{cases} 1 & \text{if } n = -1, \\ 0 & \text{if } n \neq -1. \end{cases}$$

At the end of the day, we have learned that the quantity $\bar\partial f$ plays two important roles.

(i) $\bar\partial f(a)$ is a measure of how far the image on an infinitesimal circle centered at a is itself an infinitesimal circle (at least when $\partial f(a) \neq 0$).

(ii) It is the the obstruction to the vanishing of $\int_c f(z)dz$, at least when the curve c is the (piecewise C^1) boundary of a domain.

It is natural to consider the functions for which $\bar\partial f = 0$ an interesting object. This is in fact the subject of these lectures.

2. Basics on holomorphic functions

The complex derivative is defined as the limit of a ratio like its real counterparts, and many properties likewise extend to the complex case. This seemingly innocent extension, however, has many consequences which point in the direction of the *rigidity* of functions which are differentiable in the complex sense in an open subset of the complex plane (*holomorphic maps*). Fortunately, and surprisingly, many other results say that such maps are flexible enough to perform a number of important tasks. In this section, we see several equivalent definitions of holomorphic function, and some of their most immediate consequences.

2.1. Definitions of holomorphicity based on derivatives.

2.1.1. *The complex derivative and the Cauchy-Riemann equations.*

DEFINITION 1. *Let $f : \Omega \to \mathbb{C}$, where Ω is a domain on the complex plane. Its* **complex derivative** *is*

$$f'(z) = \lim_{\mathbb{C}\ni h\to 0}\frac{f(z+h)-f(z)}{h}, \tag{2.1}$$

provided the limit exists in $\mathbb{C}$. If f has complex derivative at any point z of a domain Ω, we say that f is **holomorphic** *in Ω. The (complex) linear space of the functions which are holomorphic in Ω is denoted by $H(\Omega)$.*

Having a complex number at the denominator in (2.1) is a unique peculiarity. The fact is that $\mathbb{R}^2 \equiv \mathbb{C}$ carries a field structure, which singles it out among all $\mathbb{R}^n$, but for the familiar real line. For $n = 4$ we have the quaternions, with a skew-field structure, but attempts to exploit left/right different quotients in the quaternionic setting did not produce an equally rich theory. As we shall see below, having a complex number at the denominator (that is, having a degree of freedom in the directions to approach z) has deep consequences, the first of which are the Cauchy-Riemann equations which will appear shortly.

This is just the definition of derivative from basic calculus, and most of the properties you have seen concerning differentiable functions carry on to the complex universe, with the same proofs: derivatives of sums, products, ratios... **The fact that the product of holomorphic functions is holomorphic is of the utmost importance!** From the viewpoint of calculations, this is a trivial consequence of the definition of derivative.

EXERCISE 5. *Let f, g be holomorphic in Ω. Prove that fg is holomorphic, and that*

$$(fg)' = f'g + fg'.$$

You just have to verify that the usual proof goes through.

The closure of the holomorphic class under products will appear many times, and it is the feature that singles holomorphic functions among all other classes of solutions of (linear, homogeneous) elliptic PDEs. If you do not know what an elliptic PDE is, do not worry: here we are concerned with very specific differential equations and we will build everything from scratch.

The derivative of a composition $g \circ f$ of holomorphic functions is

$$(g \circ f)'(z) = g'(f(z))f'(z) : \tag{2.2}$$

the proof is the same as for real functions of a real variable.

EXERCISE 6. *State (2.2) in a proper manner, and prove it.*

The derivative of the inverse is more subtle, because we do not have here the notion of monotonicity. We postpone this issue to a later discussion.

An important point is that we are here differentiating in some way a function from the plane to itself, and instead of the four degrees of freedom of the Jacobian matrix, we have here two degrees of freedom because $f'(z) \in \mathbb{C} \equiv \mathbb{R}^2$. The other two degrees of freedom are eliminated by s system of partial differential equation. [1]

THEOREM 3 (Cauchy-Riemann equations). *For a map $f = u + iv : \Omega \to \mathbb{C}$ the following are equivalent.*

(i) *$f'(z)$ exists for all z in Ω.*
(ii) *f is differentiable in Ω and the* **Cauchy-Riemann equations** *hold,*

$$\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases} \tag{2.3}$$

[1] The two extra degrees of freedom appear in a number of problems, especially in geometry. They are codified in $\bar{\partial} f$, as we shall see. As the order of derivation increases, the number of the extra degrees of freedom (in the non-holomorphic case) likewise increases, and the frugality of the holomorphic world appears more and more. As it often happens, frugality is due to the rich underlying structure.

(iii) *The differential form $f(z)dz$ is closed.*

Another way to state the Cauchy-Riemann equations is

$$\bar{\partial} f = 0, \tag{2.4}$$

or

$$Jf = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}. \tag{2.5}$$

Proof. If the limit exists, f is differentiable at z because

$$f(z+h) = f(z) + f'(z)h + o(h),$$

and the expression $h \mapsto f'(z)h$ is linear from $\mathbb{R}^2$ to itself. Moreover,

$$\partial_x f(z) = \lim_{\mathbb{R} \ni \eta \to 0} \frac{f(z+\eta) - f(z)}{\eta} = f'(z) = \lim_{\mathbb{R} \ni \eta \to 0} \frac{f(z+i\eta) - f(z)}{i\eta} = -i\partial_y f(z),$$

i.e. $0 = \bar{\partial} f(z)$, and unravelling definitions this is equivalent to the Cauchy-Riemann equations. It is also equivalent to the closeness of

$$f(z)dz = (udx - vdy) + i(vdx + udy) = \alpha + i\beta :$$

closeness of α is the second equation, that of β is the first.

Suppose now that f is differentiable and it satisfies the Cauchy-Riemann equations. Then, with η, ζ real and $h = \eta + i\zeta$,

$$\begin{aligned} f(z+\eta+i\zeta) - f(z) &= \begin{pmatrix} u_x(z) & -v_x(z) \\ v_x(z) & u_x(z) \end{pmatrix} \begin{pmatrix} \eta \\ \zeta \end{pmatrix} + o(h) \\ &= u_x(z)\eta - v_x(z)\zeta + i(v_x(z)\eta + u_x(z)\xi) + o(h) \\ &= (u_x(z) + iv_x(z))(\eta + i\zeta) + o(h) \\ &= \partial_x f(z)h + o(h), \end{aligned}$$

from which we deduce that $f'(z) = \partial_x f(z)$ exists in $\mathbb{C}$. □

Corollary 1. *Let f be C^2 and holomorphic in Ω. Then, f' is holomorphic.*

Proof.

$$\bar{\partial} f' = \bar{\partial}\partial f = \partial\bar{\partial} f = 0.$$

□

For f is holomorphic, we can write

$$df(z) = f'(z)dz, \tag{2.6}$$

since $\bar{\partial} f = 0$ and $\partial f = f'$. Since we are going to integrate such expressions, it is worth spending a little time unraveling the details to see where the Cauchy-Riemann equations enter the picture from a different side.

$$\begin{aligned} f'(z)dz &= \partial_x f dz = (\partial_x u + i\partial_x v)(dx + idy) \\ &= (\partial_x u dx - \partial_x v dy) + i(\partial_x u dy + \partial_x v dx) \\ &= (\partial_x u dx + \partial_y u dy) + i(\partial_y v dy + \partial_x v dx) \\ &= du + idv = df. \end{aligned} \tag{2.7}$$

In which sense f is a potential of f'?

$$\begin{aligned} f' &= \partial_x u - i\partial_y u = \overline{\nabla u} \\ &= i(\partial_x v - i\partial_y v) = i\overline{\nabla v}, \end{aligned} \tag{2.8}$$

i.e.

$$\nabla v = i\nabla u = \nabla u \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.9}$$

where the relation between the (row) vectors is expressed in $\mathbb{C}$ and in $\mathbb{R}^2$. Equation (2.9) expresses the fact that the level curves of u and v are orthogonal, and that, after after a linear transformation, $u = \mathrm{Re} f$ is a scalar potential for f', where the latter is interpreted as a vector field, and that the same holds for $v = \mathrm{Im} f$.

Summarizing, we might view f as a vector field (as in the expression $f(z)dz$), but also as a potential (as in $df(z) = f'(z)dz$). This reflects the double nature of the complex numbers, which are scalars (a field), but can also be viewed as two dimensional vectors with real entries. The two interpretations are made compatible by the Cauchy-Riemann equations, which freeze the exponential growth of the number of partial derivatives as the order of derivation increases.

Before we draw some other consequences from theorem 3, let us consider the special form of the Jacobian: $Jf = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. We can write the matrix as

$$Jf = r \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix},$$

with $r = \sqrt{a^2+b^2} = |f'(z)|$. The matrix with the cosines and sines belongs to $SO(2)$, and in fact it represents an anti-clockwise rotation by t radians in the plane. Overall, then, $f'(z) \equiv Jf(z)$ is the composition of a dilation with center at the origin by a factor of r, and a rotation. In fact,

$$f'(z) = re^{it}.$$

The matrices representing complex derivatives are related to the complex field in several useful ways.

(i) The map $z = x + iy \mapsto \varphi(z) := \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ is a isomorphism of fields, $|z|^2 = \det \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, and $\varphi(\bar{z}) = \varphi(z)^t$ is the transpose of $\varphi(z)$.

(ii) Call ψ the identification $\psi(x + iy) = \begin{pmatrix} x \\ y \end{pmatrix}$. Then,

$$\psi(zw) = \varphi(z)\psi(w).$$

This means that in the complex derivative we divide by a matrix!

(iii) Suppose that $f'(z) = re^{it} \neq 0$. Let $c, d : [0,1] \to \Omega$ two curves starting at z, $c(0) = d(0) = z$, and let $f \circ c, f \circ d$ be their images under f. Then, the angle between $f \dot{\circ} c(0)$ and $f \dot{\circ} d(0)$ is the same as that between $\dot{c}(0)$ and $\dot{d}(0)$. In fact, the vectors tangent to the curves at z are anti-clockwise rotated by an angle of t. This is expressed by saying that f is *angle-conformal.*

(iv) Holomorphic maps are *metrically conformal.* If $f'(z) \neq 0$, then

$$\limsup_{\epsilon \to 0} \frac{\sup\{|f(z+h) - f(z)| : |h| \leq \epsilon\}}{\inf\{|f(z+h) - f(z)| : |h| \leq \epsilon\}} = 1. \tag{2.10}$$

EXERCISE 7. *Verify these assertions.*

EXERCISE 8. *Provide a proof of (2.2) using the theorem on the differentiation of a composition for several real variables, and item (i) above.*

2.1.2. *Functions which are holomorphic directly by definition and algebraic properties.*

(i) The function $f(z) = z$ is holomorphic, with derivative $f'(z) = 1$. On the contrary, $g(z) = \bar{z}$ *is not* holomorphic (e.g. because it does not satisfy the Cauchy-Riemann equations; or because you can verify by hands that the $\lim_{h\to 0} \frac{\bar{h}}{h}$ does not exist in $\mathbb{C}$). As a consequence, polynomials $p(z) = \sum_{j=0}^{n} a_j z^j$ ($a_n \neq 0$) are holomorphic in $\mathbb{C}$, and *rational functions* $r(z) = \frac{p(z)}{q(z)}$, where p, q are polynomials, and q is not the zero one, define holomorphic functions in $\mathbb{C} \setminus \{z : q(z) = 0\}$. From high-school algebra, we know that the set of the points where r is not defined consists of at most $\deg q$ points.

(ii) The *exponential function*

$$e^z := e^{x+iy} = e^x(\cos y + i \sin y) \tag{2.11}$$

defines a holomorphic function in $\mathbb{C}$. So, then, are the *complex cosine/sine functions, and the complex hyperbolic cosine/sine functions*:

$$\cos z = \frac{e^{iz} + e^{iz}}{2}, \ \sin z = \frac{e^{iz} - e^{iz}}{2i}, \ \cosh z = \frac{e^z + e^z}{2}, \ \sinh z = \frac{e^z - e^{iz}}{2}. \tag{2.12}$$

(iii) Everything we can obtain from (i), (ii), compositions, and the algebraic operations is holomorphic in its natural domain.

(iv) The *complex logarithm* (rather, its *principal branch*) $\mathrm{Log}(z)$, which we define in $\mathbb{C} \setminus (-\infty, 0]$,

$$\mathrm{Log} z = \log \sqrt{x^2 + y^2} + i\mathrm{Arg}(z), \tag{2.13}$$

and $\mathrm{Arg}(z)$ is the principal branch of the argument,

$$\mathrm{Arg}(z) = \begin{cases} \arctan(y/x) \text{ if } x > 0, \\ \arctan(y/x) + \pi \text{ if } x < 0, y > 0, \\ \arctan(y/x) - \pi \text{ if } x < 0, y < 0, \\ \pi/2 \text{ if } x = 0, y > 0 \text{ and } -\pi/2 \text{ if } x = 0, y < 0. \end{cases} \tag{2.14}$$

Once you have verified that $\mathrm{Arg}(z)$ is differentiable for $x = 0, y \neq 0$, verifying the Cauchy-Riemann equations is routine. In fact, we might define $\mathrm{Arg}(z)$ up to the negative real axis by letting $\mathrm{Arg}(x) = \pi$ for $x < 0$. Of course, with this convention $\mathrm{Arg}(z)$ is discontinuous at all negative reals. Moreover, when we learn complex numbers we meet the relations

$$e^{\mathrm{Log}(z)} = z, \text{ and } \mathrm{Log}(e^z) = z + 2\pi i m \tag{2.15}$$

for some integer m. What singles out Log among all inverses of the exponential is that $\mathrm{Log}(1) = 0$. We have

$$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}. \tag{2.16}$$

Using the logarithm, we can on the same domain define powers with complex exponent,

$$z^\alpha := e^{\alpha \mathrm{Log}(z)}, \tag{2.17}$$

which define holomorphic functions. With our choice of a definition for Log, we have that $1^\alpha = 1$. Using (2.16), we have

$$\frac{dz^\alpha}{dz} = \alpha z^{\alpha-1}, \tag{2.18}$$

as expected.

Logarithm and fractional powers are important functions, and deciding on which domains they are defined is important.

(v) Let K be a compact subset of $\mathbb{C}$, and let $F : \Omega \times K \to \mathbb{C}$ be a function such that (a) $F(\cdot, \zeta)$ is holomorphic in the open set Ω; (b) F is continuous. Let μ be a (possibly complex) Borel measure of finite variation on K. Then

$$f(z) := \int_K F(z,\zeta) d\mu(\zeta) \tag{2.19}$$

defines a function which is holomorphic in Ω. (The assumption that F be simultaneously continuous can be relaxed). To prove it, just differentiate under the integral sign (or use Morera's theorem 12 and some version of Fubini's theorem).

(vi) With the notation of (v), and $\Omega = \mathbb{C} \setminus K$,

$$\mathcal{C}\mu(z) := \frac{1}{2\pi i} \int_K \frac{d\mu(\zeta)}{z - \zeta}, \tag{2.20}$$

the *Cauchy integral* (or *Cauchy transform*) of he Borel measure μ, is holomorphic in $\mathbb{C} \setminus K$. The function $\zeta \mapsto \frac{1}{2\pi i} \frac{1}{z-\zeta}$ is the *Cauchy kernel*.

2.1.3. *Real and imaginary part of a holomorphic function.* Let $f = u + iv$ be holomorphic in a domain Ω, and suppose that $f \in C^2$ (we will show that in fact $f \in C^\infty$). The *Laplacian* of u is

$$\Delta u = \partial_{xx} u + \partial_{yy} u.$$

We say that u is *harmonic* in Ω if $\Delta u = 0$ there.

Proposition 1. *Let $f = u + iv$ be holomorphic and C^2. Then, u and v are harmonic.*

Proof. By the Cauchy-Riemann equations,

$$\partial_{xx} u + \partial_{yy} u = (v_y)_x - (v_x)_y = 0,$$

and similarly for v. □

We say that the harmonic function v is *conjugate* to the harmonic function u if $u + iv$ is holomorphic, or that v is the *conjugate harmonic function* of u. Since if is holomorphic if and only if f is, if v is conjugate to u, then u is conjugate to $-v$.

Proposition 2. *If v_1, v_2 are conjugate harmonic to the same harmonic function u on a domain Ω, then $v_2 - v_1$ is constant.*

Proof. By the Cauchy-Riemann equations, $\nabla v_1 = \nabla v_2$. □

Harmonic functions play an important role in the theory, and are an important object in themselves. The theory of harmonic functions extends to $\mathbb{R}^n$ for any $n \geq 2$. The advantage of dimension $n = 2$ is that we can use holomorphic theory to study them.

The Laplace operator can be expressed in terms of complex derivatives,

$$\Delta = 4\partial\bar{\partial} = 4\bar{\partial}\partial, \tag{2.21}$$

since

$$4\bar{\partial} = (\partial_x - i\partial_y)(\partial_x + i\partial_y) = \partial_x\partial_x - i^2\partial_y\partial_y = \Delta.$$

A rather surprising consequence of (2.21) is that the Laplace operator is invariant under composition with holomorphic maps, in the sense that

$$\Delta(u \circ f) = [(\Delta u) \circ f]|f'|^2, \tag{2.22}$$

if f is C^2. Since $\bar{\partial} f = 0$, in fact,

$$\bar{\partial}(\partial(u \circ f)) = \bar{\partial}[((\partial u) \circ f)f'] = ((\bar{\partial}\partial u) \circ f)\overline{f'}f'.$$

As a consequence,

Proposition 3. *If $f : A \to \Omega$ is holomorphic, and $u : \Omega \to \mathbb{C}$ is harmonic, then $u \circ f$ is harmonic.*

The function $u(z) = \log|z| = \frac{1}{2}\log|z|^2$ is harmonic on $\mathbb{C} \setminus \{0\}$. This would follow from the fact that $\log z = \log|z| + i \arg z$ can be defined on $\mathbb{C} \setminus L$, where L is a half-line starting at 0, and it is holomorphic there, but also by direct calculation:

$$\bar{\partial}\partial \log(\bar{z}z) = \bar{\partial}\left(\frac{\partial(z\bar{z})}{\bar{z}z}\right) = \bar{\partial}\left(\frac{\bar{z}}{\bar{z}z}\right) = \bar{\partial}\left(\frac{1}{z}\right) = 0.$$

Corollary 2. *If $f : \Omega \to \mathbb{C}$ is holomorphic, then $\log|f(z)|$ is harmonic on $\Omega \setminus \{z : f(z) = 0\}$.*

The example of the logarithm $u(z) = \log|z|$ also shows that the conjugate harmonic function of a given harmonic $u : \Omega \to \mathbb{R}$ does not always exist. If, as above, we remove a half-line L starting at 0, then $v(z) = \arg z + k$ is a conjugate harmonic function there, but it can not be extended to the whole $\mathbb{C} \setminus \{0\}$. If a global conjugate harmonic w existed, by uniqueness it would have the same form of v on $\mathbb{C} \setminus L$ for some k, but then it could not be extended to a continuous function on $\mathbb{C} \setminus \{0\}$.

We shall see shortly, however, that a conjugate harmonic function always exists *locally*, and it exists globally if the domain has the Volterra-Poincaré property.

2.1.4. *Fractional linear maps.* We consider a special, yet useful and interesting class of holomorphic functions.

A *fractional linear map* is a function having the form

$$\varphi(z) = \frac{az + b}{cz + d}, \tag{2.23}$$

where $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$. In fact, dividing all entries by the same complex number, the corresponding map φ remains the same. We might then suppose that $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$, i.e. that the matrix belongs to the *special linear group* $SL(2, \mathbb{C})$. From a different viewpoint, we might identify $SL(2, \mathbb{C}) \equiv PGL(2, \mathbb{C})$, where the latter is the *projective linear group* acting on the *complex projective line* $\mathbb{C}P^1$. We will consider this point of view in the section on the Riemann sphere.

The domain of φ is $\mathbb{C} \setminus \{-d/c\}$, if $c \neq 0$. However, $\mathbb{C}$ is not the best universe where studying maps of the form (2.23). We consider the *one point compactification*

$\mathbb{C}_*$ of the complex plane by adding a point ∞. We make $\mathbb{C}_*$ into a topological space extending Euclidean $\mathbb{C}$ by adding a basis of neighborhoods for ∞,

$$N(\infty, R) = \{z : |z| > R\}.$$

The topological space $\mathbb{C}_*$ is called *complex projective*, *Riemann sphere*, or *extended plane*, depending of which of its features one is mostly interested in. We extend φ to a map $\varphi : \mathbb{C}_* \to \mathbb{C}_*$, by setting

$$\begin{cases} \varphi(-d/c) = \infty \text{ and } \varphi(\infty) = a/c \text{ if } c \neq 0, \\ \varphi(\infty) = \infty \text{ if } c = 0. \end{cases} \tag{2.24}$$

We denote by $\mathcal{M}(\mathbb{C}_*) \equiv SL(2,\mathbb{C}) \equiv PGL(2,\mathbb{C})$, the *linear group on* $\mathbb{C}_*$, the family of the fractional linear transformations.

We would like to have a notion of "derivative at infinity" which allows us to measure angles between curves going to ∞. We will introduce a geometric definition when discussing the stereographic projections, which literally identify the extended plane with a sphere (the Riemann sphere, in fact). Here we are satisfied with a less geometric (but justifiable in terms of differentiable topology, introducing charts at 0 and at ∞) definition.

DEFINITION 2. *Suppose that f maps an open subset Ω of $\mathbb{C}_*$ to $\mathbb{C}_*$ is continuous, and holomorphic on $\Omega \setminus (\{\infty \cup \{z \in \Omega : f(z) = \infty\})$.*

(i) *If $\infty \in \Omega$ and $f(\infty) \neq \infty$, we say that f is holomorphic at ∞ if and only if $f(1/z)$ is holomorphic at 0 (where by definition $f(1/0) = f(\infty)$). We let $f'(\infty) = \frac{d}{dz}|_{z=0} f(1/z)$.*

(ii) *If $\infty \in \Omega$ and $f(\infty) = \infty$, we say that f is holomorphic at ∞ if and only if $1/f(1/z)$ is holomorphic at 0 (where by definition $f(1/0) = f(\infty)$). We let $f'(\infty) = \frac{d}{dz}|_{z=0}\left(\frac{1}{f(1/z)}\right)$.*

(iii) *If $\infty \neq a \in \Omega$ and $f(a) = \infty$, we say that f is holomorphic at a if and only if $1/f$ is holomorphic at a. We let $f'(a) = \frac{d}{dz}|_{z=a}\left(\frac{1}{f(z)}\right)$.*

For instance, if $a \neq 0$,

$$\frac{d(az+b)}{dz}|_{z=\infty} = \frac{d}{dw}|_{w=0}\left(\frac{1}{a/w+b}\right) = \left(\frac{(a+bw)-wb}{(a+bw)^2}\right)_{w=0} = \frac{1}{a} :$$

if $|a| = |f'(0)|$ is large, then points move far from 0; which means that they move towards ∞. From the viewpoint of ∞, in fact, we see $|f'(\infty)|$ large.

An interpretation of these definitions is that when $\zeta \to \infty$, $1/\zeta \to 1/\infty = 0$, and this suggests to define a "difference at ∞" as $\frac{1}{\zeta} - 1\infty = 1/\zeta$. The difference quotient at ∞ when $f(\infty) = L \neq \infty$, for instance, becomes

$$\frac{f(\zeta) - f(\infty)}{1/\zeta - \infty} = \zeta(f(\zeta) - f(\infty)),$$

hence,

$$\lim_{\zeta\to\infty} \frac{f(\zeta) - f(\infty)}{1/\zeta - \infty} = \lim_{\zeta\to\infty} \zeta(f(\zeta) - f(\infty) = \lim_{w\to 0} \frac{f(1/w) - f(1/0)}{w} = \frac{d}{dw}|_{w=0}[f(1/w)].$$

which is definition (i).

EXERCISE 9. *Find analogous interpretations for (ii) and (iii).*

It will turn out that these definitions are the right ones when we want to expand a function as a Taylor series at ∞, when we compute residues, and so on.

We enumerate the basic geometric facts about fractional linear transformations.

PROPOSITION 4. (i) *The map* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$ *is a group isomorphism from* $SL(2,\mathbb{C})$ *onto* $\mathcal{M}(\mathbb{C}_*)$. *In particular, fractional linear maps are bijective on* $\mathbb{C}_*$.

(ii) *Any map* φ *in* $\mathcal{M}(\mathbb{C}_*)$ *can written as the composition of (at most four) maps of the form*

$$\begin{aligned} z &\mapsto 1/z, && \text{the inversion in the unit circle,} \\ z &\mapsto z+\alpha, && \text{a translation by } \alpha \in \mathbb{C}, \\ z &\mapsto \beta z, && \text{a rotation-dilation by } \beta = re^{it} \in \mathbb{C}\setminus\{0\}. \end{aligned} \tag{2.25}$$

(iii) *Let* z_0, z_1, z_∞ *be three distinct points in* $\mathbb{C}_*$. *Then there exists a unique fractional linear transformation such that* $z_0 \mapsto 0$, $z_1 \mapsto 1$, $z_\infty \mapsto \infty$. *In fact, the map*

$$\varphi(z) = \frac{z-z_0}{z-z_\infty} : \frac{z_1-z_0}{z_1-z_\infty} \tag{2.26}$$

has this property (with the convention that in this case $\infty/\infty = 1$*).*

(iv) *Linear fractional transformations map the set of straight lines and circles into itself.*

(v) *Let* (a_1,a_2,a_3), (b_1,b_2,b_3) *be triples of disjoint points in* $\mathbb{C}_*$. *There is a unique linear fractional function mapping* a_j *to* b_j $(j=1,2,3)$.

PROOF. (i) can be checked by direct calculation; (ii) is an easy exercise; (iii) consists in verifying that (2.26) does the job; (v) is an immediate consequence of (iv) and the fact that $\mathcal{M}(\mathbb{C}_*)$ is a group. We only have to check (iv), i.e. that $z \mapsto 1/z$ maps straight lines and circles to straight lines and circles. We verify it directly. Any straight line can be written in the form

$$\alpha\frac{z+\bar z}{2} + \beta\frac{z-\bar z}{2i} + \gamma = 0, \tag{2.27}$$

with real α,β,γ, $(\alpha,\beta)\neq(0,0)$. Replacing $z = 1/w$,

$$\begin{aligned} 0 &= |w|^2\left(\alpha\frac{1/w+1/\bar w}{2} + \beta\frac{1/w-1/\bar w}{2i} + \gamma\right) \\ &= \alpha\frac{w+\bar w}{2} - \beta\frac{w-\bar w}{2i} + \gamma|w|^2, \end{aligned} \tag{2.28}$$

which is the equation of a circle passing through the origin, if $\gamma\neq 0$, or of a straight line if $\gamma = 0$. On the other hand, all circles have an equation similar to (2.28) (with respect to the variable w),

$$\delta + \alpha\frac{w+\bar w}{2} - \beta\frac{w-\bar w}{2i} + |w|^2 = 0$$

(with appropriate conditions on the reals α,β,δ), and $z = 1/w$ maps them to

$$\delta|z|^2 + \alpha\frac{z+\bar z}{2} + \beta\frac{z-\bar z}{2i} + 1 = 0,$$

which is the equation of a circle ($\delta\neq 0$) or a straight line ($\delta\neq 0$), which do not pass through the origin. □

EXERCISE 10. *Fill in the details in the proof of proposition 4. Also, find conditions on circles and lines so that $1/z$ maps lines to lines, lines to circles, etcetera.*

The Möbius group $\mathcal{M}(\mathbb{C}_*)$ has *six* real degrees of freedom. In other terms, it is a *Lie group* having dimension 6. No Riemannian metric on $\mathbb{C}_*$ can be invariant under the full group, since the isometry group of a two-dimensional manifold has real dimension ≤ 3. All maps in $\mathcal{M}(\mathbb{C}_*)$, however, are *conformal* (we introduce the notion of holomorphicity at ∞ just below). We will see later that, in fact, $\mathcal{M}(\mathbb{C}_*)$ exhausts the *conformal group* of $\mathbb{C}_*$.

2.1.5. *Mapping properties of some holomorphic functions.* A function $f : \Omega \to \mathbb{C}$ defined on a domain Ω is *conformal* if it is holomorphic and it a bijection onto $f(\Omega)$, with a holomorphic inverse. We will see that, $f(\Omega)$ is open (if f is not a constant) and, if f is injective, the inverse map is holomorphic.

(i) The fractional linear transformations are conformal (excluding for the moment ∞ from domain and co-domain), and we already know quite a bit about them. An especially interesting one is the *Cayley map*, which sends the *upper half plane* $\mathbb{C}_+ = \{z = x + iy : y > 0\}$ onto the unit disc $\mathbb{D} = \{w : |w| < 1\}$. There are different tastes for the definition. Say we want $i \mapsto 0$, $0 \mapsto 1$, $\infty \mapsto -1$, which gives us
$$z \mapsto \frac{i - z}{i + z} = w.$$

(ii) The function $z \mapsto 1/z$ maps the unit disc in the interior of its complement, $\mathbb{D}$ to $\mathbb{C}_* \setminus \mathrm{cl}\mathbb{D}$, in such a way 0 goes to ∞. The boundary of the unit disc is mapped onto itself, $1/e^{it} = e^{-it} = \overline{e^{it}}$. If you tink of it, this is interesting: the function $z \mapsto \bar{z}$ is not holomorphic, but its values on the boundary of the unit disc extend to the function $z \mapsto 1/z$, which is holomorphic in the exterior of the unit circle, and on the disc, provided 0 is removed.

(iii) The exponential map $z \mapsto e^z$ is $2\pi i$ periodic. It conformally maps the strip $S = \{z = x + iy : -\pi < y < \pi\}$ to the complex plane with the negative real axis removed. By periodicity, $\lim_{y\to\pi} e^{x+iy} = \lim_{y\to-\pi} e^{x+iy}$. As y moves across the real line, we see that the complex plane is mapped $\infty - 1$ onto $\mathbb{C} \setminus \{0\}$, the *punctured disc.*

The lines $y = d$ are mapped $1 - 1$ to rays starting at the origin, while the lines $x = c$ are mapped $\infty - 1$ to circles centered at the origin.

The inverse map of the exponential is (some version of) the logarithm.

(iv) A perturbation of $z \mapsto z^2$. Consider the map $z \mapsto z^\alpha = f(z)$, with $\alpha > 0$, which (for non-integer α) is only defined on a portion of the plane, say $\mathbb{C} \setminus (-\infty, 0]$. Then, f maps $L_\theta = \{re^{i\theta} : r \geq 0\}$ to $L_{\alpha\theta}$. When α is integer, the map can be defined on $\mathbb{C}$, and it is a $\alpha - 1$ holomorphic map, but for the origin, which is the only solution of $z^\alpha = 0$.

(v) Consider the function $w = f(z) = z^2 - 1$. By the properties of the complex roots, is is $2 - 1$ but at the origin (where if fact its derivative vanishes). For $w \neq -1$ fixed, in fact, there are two distinct solutions $\pm z$. That means that, no matter how you partition $\mathbb{C} \setminus \{0\}$ in two regions (not open, of course) which are mapped one into the other by $z \mapsto -z$, you have one and only one element of $f^{-1}(w)$ $(w \neq -1)$ in each. Best for visualization is partitioning along the imaginary axis. You will see that, as a map, $z^2 - 1$ does not behave too differently from z^2.

EXERCISE 11. *Show that, after a linear change of coordinates in the z and in the w plane, any 2^{nd} degree polynomial $w = az^2 + bz + c$ can be written in the form $w = z^2$ or in the form $w = z^2 - 1$.*

2.2. Complex integration and the local Cauchy theorem: the C^1 case. If the holomorphic function f is also C^1, we can use a number of important tools: the Volterra-Poincaré theorem on conservative fields, and Pompeiu formula. In this subsection we are going to do this, and a notable consequence of this is another fundamental characterization of holomorphic functions: those which can be locally expanded in power series. Right after, we will see that the C^1 requirement can in fact be dropped.

We will see a couple of versions of *Cauchy integral theorem*, which is one of the foundational results of holomorphic function theory. Comparing their hypothesis, you might guess that there is a meta-Cauchy theorem of which these results are special cases. This is correct, but you will have to wait.

THEOREM 4 (The C^1, local case of the Cauchy theorem). *Let Ω be a domain in the complex plane, and $f : \Omega \to \mathbb{C}$ be a C^1 map. Then, f is holomorphic if and only if for point a in Ω there is a disc $D(a, \delta)$ in Ω such that*

$$\int_c f(z)dz = 0 \tag{2.29}$$

for all closed, piecewise C^1, closed curves in $D(a, \delta)$.

PROOF. We saw that f is holomorphic if and only if the 1-form $f(z)dz$ is closed. By the Volterra-Poincaré theorem, then, $\int_c f(z)dz = 0$ for all closed curves in $D(a, \delta)$. Viceversa, suppose that f is continuous and that $\int_c f(z)dz = 0$ for all curves c in $D(a, \delta)$. Let $f(z)dz = \alpha + i\beta$ the decomposition of the 1-form in its real and imaginary parts. Again by the Volterra-Poincaré theorem, α and β are conservative 1-forms in $D(a, \delta)$ and potentials for them are provided by

$$U(z) = \int_a^z \alpha = \int_a^z (udx - vdy), \; V(z) = \int_a^z \beta = \int_a^z (vdx + udy).$$

Then,

$$U(z) + iV(z) = F(z) := \int_a^z f(w)dw$$

satisfies the equations

$$\begin{cases} U_x &= u, \\ U_y &= -v, \\ V_x &= v, \\ V_y &= u. \end{cases}$$

Since F is C^2 and it satisfies the Cauchy-Riemann equations, it is holomorphic in $D(a, \delta)$. Moreover, $F' = U_x + iV_x = u + iv = f$, and we have seen in corollary 1 that the derivative of a holomorphic function is holomorphic, if it is differentiable. □

The proof of theorem 4 also gives the following version of the fundamental theorem of calculus.

Corollary 3 (First fundamental theorem of calculus). *Suppose that f is continuous in a domain Ω which satisfies the Volterra-Poincaré property, and fix a in Ω. Then,*

$$F(z) = k + \int_a^z f(w)dw$$

defines a holomorphic function F on Ω and $F' = f$,

$$\frac{d}{dz}\int_a^z f(w)dw = f(z).$$

The other fundamental theorem of calculus holds without restrictions on the domain.

Proposition 5 (Second fundamental theorem of calculus). *Let f be holomorphic and C^2 in a domain Ω. Then,*

$$f(b) - f(a) = \int_a^b f'(z)dz = \int_\gamma f'(z)dz, \tag{2.30}$$

where the complex integral can be taken along any curve γ from a to b.

Proof. We know by corollary 1 that f' is holomorphic, and that f is its potential in the sense of (2.8). The relation (2.30) easily follows. □

The C^2 assumption is pleonastic: we will show that f holomorphic $\implies f \in C^\infty$. Using the same ideas, we can prove the existence of the conjugate harmonic function on special domains.

Theorem 5 (Existence of the conjugate harmonic function). *Let $u : \Omega \to \mathbb{R}$ be harmonic, and suppose Ω satisfies the Poincaré-Volterra property. Then, u has a harmonic conjugate in Ω.*

In particular, a conjugate harmonic function exists locally.

Proof. We wish to find v such that the Cauchy-Riemann equations hold, $v_x = -u_y$, $v_y = u_x$. The 1-form $\omega = -u_y dx + u_x dy$ is irrotational because u is harmonic, $(-u_y)_y = (u_x)_x$, hence, there exists v such that $dv = \omega$, which are in fact the Cauchy-Riemann equations. □

We close this subsection with a key result.

Theorem 6 (Cauchy formula for C^1 holomorphic functions). *Let f be a C^1 holomorphic function in Ω, let A be a bounded domain with closure in Ω and piecewise C^1 boundary $c = \partial A$. For a in A,*

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(w)}{w-a}dw. \tag{2.31}$$

Proof. It is an immediate consequence of Pompeiu's formula. □

We can differentiate 2.31 under the integral, and we obtain Cauchy-type formulas for all derivatives.

Corollary 4. *With the same hypothesis of theorem 6, we have*

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_c \frac{f(w)}{(w-a)^{n+1}}dw. \tag{2.32}$$

Pompeiu's formula gives us another version of the Cauchy theorem.

THEOREM 7 (Cauchy theorem for boundaries). *Let f be a C^1 holomorphic function in Ω, let A be a bounded domain with closure in Ω and piecewise C^1 boundary $c = \partial A$. Then,*

$$\int_c f(z)dz = 0.$$

In fact, the conclusion holds, for the same reason, under the more general assumption that f if holomorphic in A, continuous in cl$A$, and A has piecewise C^1 boundary.

Before we proceed, we observe that **being holomorphic is much more than being smooth in some sense: the values of f on the boundary of the domain completely determine those in the interior!** This imposes a great degree of rigidity on holomorphic functions, and one might wonder if they come in great supply. "Forcing" holomorphic functions to perform given tasks typically requires imagination and hard work, and related open problems still abound. We will see some instances of this during the course. Right now, we draw a simple consequence from (2.31).

COROLLARY 5 (Mean Value Property). *Let f be holomorphic and C^1 in Ω, and let the closure of $D(a, r)$ be contained in Ω. Then,*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})dt. \tag{2.33}$$

PROOF. This is just Cauchy formula, with the parametrization $w = a+re^{it}$. □

EXERCISE 12 (Change of variable formulas for holomorphic line integrals). *Let f be holomorphic in Ω, g be holomorphic in a region containing $f(\Omega)$, and c be a curve in Ω. Then,*

(i) $\frac{d}{dt}(f \circ c)(t) = f'(c(t))c'(t)$ *(where the product is that in $\mathbb{C}$);*
(ii) $\int_{f(c)} g(w)dw = \int_c g(f(c(t)))c'(t)dt$.

2.2.1. *Holomorphic functions which are defined by line integral.* Given a holomorphic function f on Ω, k complex, and $a \in \Omega$, we use the fundamental theorem of calculus 3 to construct a new function

$$F(z) = k + \int_a^z f(\zeta)d\zeta$$

provided $f(\zeta)d\zeta$ is conservative. In particular, we can do it when Ω satisfies the Volterra-Poincaré property. Let's consider some examples.

(i) $f(z) = \frac{1}{z^2}$ has domain $\mathbb{C}\setminus\{0\}$, which *does not* satisfy the Volterra-Poincaré property; nonetheless $g(z) = \int_1^z \frac{d\zeta}{\zeta^2}$ is well defined. In fact, $g(z) = 1 - \frac{1}{z}$, which is holomorphic in $\mathbb{C} \setminus \{0\}$, $g'(z) = \frac{1}{z^2}$, and $g(1) = 0$.
(ii) A very different story is that of $f(z) = \frac{1}{z}$, as you can guess. If we restrict the domain to make it Volterra-Poincaré, for instance $\mathbb{C} \setminus (-\infty, 0]$, then $g(z) = \int_1^z \frac{d\zeta}{\zeta}$ is well defined. You are invited to prove that $g(z) = \mathrm{Log}(z)$ is the principal branch of the logarithm we defined earlier.

We will meet again this integral, and what now seem disappointing features will turn out to be a formidable tool.

Using line integrals we can define (with care) the logarithm(s) of a holomorphic function. It is tempting using (ii):

$$\log(f(z)) = \log(f(a)) + \int_{f(a)}^{f(z)} \frac{dw}{w},$$

where f is holomorphic on a Volterra-Poincaré domain Ω, $a \in \Omega$, $f(z) \neq 0$ in Ω, and log can be chosen, e.g., to be the principal determination of the logarithm. There is however an important obstacle: $f(\Omega)$ might *not be* Volterra-Poincaré, and the integral on the right might not be well defined. However, after a change of variables $w = f(\zeta)$, which we can at least formally do, the right hand side becomes

$$\log(f(a)) + \int_{f(a)}^{f(z)} \frac{dw}{w} = \int_a^z \frac{f'(\zeta)}{f(\zeta)} d\zeta,$$

which is well defined. The idea is to use the last expression as the definition. Does it have the properties one expects from the logarithm? Yes.

THEOREM 8 (Existence of the logarithm of a function). *Let Ω be a Volterra-Poincaré domain, a a point in Ω, f holomorphic in Ω with no zeros. Define*

$$h(z) = \log(f(a)) + \int_a^z \frac{f'(\zeta)}{f(\zeta)} d\zeta, \tag{2.34}$$

where $\log(f(a)) = \log|f(a)| + i\arg(f(a))$*, and one of the possible arguments has to be chosen.*

Then, (i) $e^{h(z)} = f(z)$*; (ii)* $h(a) = \log(f(a))$.

Moreover, if k is another function satisfying (i), then $k(z) - h(z) = 2\pi i m$ for some integer m.

The theorem and its consequences are of interest even in the case $f(z) = z$.

PROOF. We hope $f(z)e^{-h(z)}$ turns out to be constant. As a matter of fact,

$$\frac{d}{dz}(f(z)e^{-h(z)}) = f'(z)e^{-h(z) - f(z)e^{-h(z)}} h'(z) = e^{-h(z)} \left(f'(z) - f(z)\frac{f'(z)}{f(z)} \right) = 0.$$

Then, $f(z) = Ce^{h(z)}$, where the value of $C = 1$ since $f(a) = Ce^{\log(f(a))} = Cf(a)$.

If k has property (i), then $k'(z)f(z) = f'(z)$, and by the fundamental theorem of calculus, $k(z) = k(a) + \int_a^z \frac{f'(\zeta)}{f(\zeta)} d\zeta$, and $e^{k(a)} = f(a) = e^{h(a)}$, hence $k(a) = 2\pi i m + h(a)$ for some integer m. □

EXERCISE 13. *Find a Poincaré domain Ω and a function f which is holomorphic on Ω such that $f(\Omega)$ is not Volterra-Poincaré.*

We denote by $\log f(z)$ the function h defined by (2.34), keeping in mind that each time we have to make a choice for the meaning of $\log f(a)$, Having defined the complex logarithm, we can define the fractional powers. For any complex α, and f, Ω as in theorem (8), let

$$f(z)^\alpha = e^{\alpha \log f(z)}. \tag{2.35}$$

Beware! Here, too, we have to make a choice for $\log f(a)$.

EXERCISE 14. (i) *Show that, if $\alpha = m/n$ with m, n positive integers, relatively prime, then (2.35) defines exactly n functions $f(z)^{m/n} = \sqrt[n]{f(z)^m}$ on Ω. Also, show that $\sqrt[n]{f(z)^n} = f(z)$, as one expects.*

(ii) *For any complex α, show that* $\frac{df(z)^\alpha}{dz} = \alpha f(z)^{\alpha-1} f'(z)$.

We have a special interest in $m/n = 1/2$, because will be used in the proof of Riemann's mapping theorem. As part of the proof, we will show that $\sqrt{f(z)}$ is in fact a conformal map.

2.3. Expansion of a holomorphic function as a power series.

2.3.1. *Holomorphic functions can be expanded as power series.*

THEOREM 9 (Expansion of a holomorphic function as a power series). *Let f be a holomorphic, C^1 function in Ω, a a point in Ω, and $D(a, r)$ a disk contained in Ω and let $c = \partial D(a, r)$. Then,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \tag{2.36}$$

converges in $D(a, r)$, where

$$a_n = \frac{1}{2\pi i} \int_c \frac{f(w)}{(w-a)^{n+1}} dw. \tag{2.37}$$

In particular, f is C^∞ and

$$a_n = \frac{f^{(n)}(a)}{n!} \tag{2.38}$$

is independent of r.

PROOF.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_c \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \int_c \frac{f(w)}{(w-a)-(z-a)} dw \\ &= \frac{1}{2\pi i} \int_c \frac{f(w)}{1-\frac{z-a}{w-a}} \frac{1}{w-a} dw \\ &= \frac{1}{2\pi i} \int_c \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} f(w) dw \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_c \sum_{n=0}^{\infty} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n. \end{aligned}$$

We can exchange integral and series because the series converges totally on the curve. □

Actually, there is no reason to integrate on a circle centered at a: we use here (i) Cauchy formula, and (ii) the fact that for w on c we have $|z-a| < |w-a|$. It suffices, that is, that z belongs to a disc centered at a which lies in the interior of c (e.g. the disc is contained in a region A with closure inside Ω, and $c = \partial A$ is C^1). Not even this is necessary, however. We will see this after we prove the global Cauchy theorem

2.3.2. *Power series are holomorphic.* Viceversa, power series define holomorphic functions in within their (open) disc of convergence.

THEOREM 10 (Power series are holomorphic). *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n. \tag{2.39}$$

Then, f defines a holomorphic function in $D(a, r)$, where the **radius of convergence** *r is given by* **Cauchy-Hadamard's formula***:*

$$r = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}. \tag{2.40}$$

In fact,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}. \tag{2.41}$$

The series (2.39) converges **totally** *on cl$(D(a,\rho))$ for each $\rho < r$:*

$$\sum_{n=0}^{\infty} |a_n|\rho^n < \infty. \tag{2.42}$$

The series diverges for $|z-a| > r$.

The proof is the same as that of the corresponding statement for real power series which is given in Advanced Calculus. We sketch it here for completeness, since it is basically a simple comparison with a geometric series.

Proof. Without loss of generality $a = 0$. If $\rho < r$ and $\lambda > 1$ is fixed, then $|a_n|\rho^n \le \frac{\lambda^n \rho^n}{r^n}$ for n large enough. If we choose $\lambda < r/\rho$, the series $\sum_{n=0}^{\infty} |a_n|\rho^n$ converges. i.e. the power series (2.39) converges *totally* in cl$(D(a,\rho))$, and, in particular, $f(z)$ converges absolutely and uniformly there. Similarly we can prove that (2.39) diverges if $|z-a| > r$, because in this case $a_n(z-a)^n$ does not tend to 0 as $n \to \infty$.

The series of the derivatives in (2.41) has the same radius of convergence of $f(z)$, hence it converges uniformly in $D(a,\rho)$ for all $\rho < r$. By a theorem of Advanced Calculus, (2.41) holds. □

Theorems 9 and 10 have an immediate, important consequence. The function f defined by the power series (2.39) can be extended to a holomorphic function on $D(a,r)$, but not on $D(0,R)$ with $R > r$. If it were, in fact, by theorem 9 f could be expanded as a power series on the larger disc, but this would contradict the fact that the radius of convergence of the series is r.

In the other direction, using theorem 9, then 10, we can extend a function f which is holomorphic in a domain Ω to a holomorphic f_1 in $\Omega \cup D(a,r)$. This procedure is called *analytic continuation* of f. There might be obstructions, since f_1 might have two different values at some points of Ω. A good example is Log$z$, the principal determination of the complex logarithm, which can not be continued across the negative real axis, due to the jump discontinuity of Arg.

There are two ways to go around this. The conservative one is to consider limit the extension to $\Omega \cup A$, where $A \subseteq D(a,r)$ is chosen in a way that $\Omega \cup A$ is still connected and that we do not have multiple values for f. The progressive way is to consider multi-valued holomorphic functions or, even better, to redefine their domain to be more general than a planar domain. This second route leads to *Riemann surfaces*.

2.3.3. *Power series expansion of holomorphic functions.* Here are some power expansions of holomorphic functions.

(i) For $f(z) = e^z$, we have $f^{(n)}(z) = e^z$, hence $f^{(n)}(0) = 1$ and

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

which converges on $\mathbb{C}$.

(ii) We have already met the *geometric series*,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$

when $|z| < 1$. When you proved the expansion in Calculus, it was part of the statement that the radius of convergence is 1. Since the singularity of $f(z) = \frac{1}{1-z}$ which is closest to 0 is $z = 1$, however, the fact that the radius of convergence of the series is 1 is something we know *a priori*.

The same function can be expanded in series, say, around $z = i$, and the radius of convergence is $|1 - i| = \sqrt{2}$. How do we find the Taylor coefficients? With $w = z - i$,

$$\begin{aligned} \frac{1}{1-z} &= \frac{1}{1-i-w} \frac{1}{1-i} \frac{1}{1-\frac{w}{1-i}} \\ &= \sum_{n=0}^{\infty} \frac{w^n}{(1-i)^{n+1}} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}. \end{aligned}$$

Exercise 15. *Use the series expansion of e^z and properties of the binomial coefficients to show that $e^{z+w} = e^z e^w$.*

From now on we consider series centered at $z = 0$. We denote here by $r(f)$ the radius of convergence of the Taylor series of f at $z = 0$. It is clear that

$$r(f + g), r(fg) \geq \min(r(f), r(g)),$$

where $>$ can occur if some singularity is canceled in the sum (as in $1/(1 - z) - 1/(1 - z)$) or in the product (as in $1/(1 - z) \cdot (1 - z)$).

Proposition 6. *Let $a_n(f)$ be the n^{th} coefficient of the series expansion of f at $z = 0$. Then,*

$$\begin{aligned} a_n(f + g) &= a_n(f) + a_n(g), \\ a_n(cf) &= ca_n(f), \\ a_n(fg) &= \sum_{i+j=n} a_i(f)a_j(g) \text{ (the \textbf{convolution} of } \{a_n(f)\} \text{ and } \{a_n(g)\}), \\ a_n(f') &= (n + 1)a_{n+1}(f), \\ a_n(1) &= \delta_0(n) = \begin{cases} 1 \text{ if } n = 0, \\ 0 \text{ if } n > 0 \end{cases}, \\ a_n(zf(z)) &= a_{n-1}(f) \text{ where, by default, } a_0(zf(z)) = 0, \\ a_n\left(\frac{f(z) - f(0)}{z}\right) &= a_{n+1}(f). \end{aligned}$$

Exercise 16. *Verify all the items in proposition 6.*

EXERCISE 17. *Suppose $f(0)=0$. Write the expression of $a_n(g\circ f)$ in terms of $\{a_n(g)\}$ and $\{a_n(f)\}$. Provide an estimate for the radius of convergence. Write a relation relating $\{a_n(g)\}$ and $\{a_n(f)\}$ when $g(f(z))=z$ (inverse mapping theorem, as it was approached by Lagrange, see* Lagrange inversion theorem on wikipedia *for the whole story).*

The last two items are the reason why, thinking of series coefficients rather that values of the function, the multiplication operator $f(z)\mapsto zf(z)$ is often called the *(forward) shift*, while $f(z)\mapsto \frac{f(z)-f(0)}{z}$ is called the *back-shift*.

These properties are not just useful in calculations, as we shall see shortly, but they also provide important links between holomorphic theory and combinatorics (the craft of *generating functions*), Fourier series, linear difference equations, and control theory.

Let's see some examples.

(i) We want to compute a closed form for $f(z)=\sum_{n=0}^{\infty}\frac{z^n}{n+1}$. We observe that

$$\begin{aligned}\frac{d}{dz}(zf(z)) &= \frac{d}{dz}\sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1}\\ &= \sum_{n=0}^{\infty}z^n=\frac{1}{1-z}\\ &= -\frac{d}{dz}\log(1-z),\end{aligned}$$

where we take log to be the principal determination of the logarithm, $\log(1)=0$. We have then:

$$f(z)=\frac{1}{z}\log\frac{1}{1-z}\ \text{(with } f(0)=1).$$

Also, $r(f)=1$.

(ii) Suppose that, on the contrary, we want to expand $f(z)=\frac{1}{(1-z)^2}$. Since $f(z)=\frac{d}{dz}\frac{1}{1-z}$, we have $a_n(f)=(n+1)a_{n+1}(1/(1-z))=n+1$, then

$$\frac{1}{(1-z)^2}=\sum_{n=0}^{\infty}(n+1)z^n.$$

(iii) [*Fibonacci-type sequences*] Consider a sequence $\{a_n\}$, where $a_0=p$ and $a_1=q$ are known, and

$$a_{n+1}=a_n+a_{n-1}. \tag{2.43}$$

Fibonacci corresponds to $p=0$, $q=1$.

We wish to better understand how the sequence terms behave, maybe finding an expression in closed for the sequence. We start writing the *generating function* $f(z)$ for the sequence,

$$f(z)=\sum_{n=0}^{\infty}a_nz^n=p+qz+\sum_{n=2}^{\infty}a_nz^n.$$

What does the recursive relation (2.43) tell us? If we multiply both sides by z^n and sum for $n \geq 1$, the left hand side becomes:

$$\sum_{n=1}^{\infty} a_{n+1} z^n = \frac{1}{z}[f(z) - p - qz].$$

The right hand side is

$$\sum_{n=1}^{\infty} (a_n + a_{n-1}) z^n = f(z) - p + zf(z).$$

Comparing, we have

$$f(z) - p - qz = z(f(z) - p + zf(z)),$$

i.e.

$$f(z) = \frac{(p-q)z - p}{z^2 + z - 1}. \tag{2.44}$$

For simplicity, consider here p, q real. The roots of the denominator are $z_\pm = \frac{-1 \pm \sqrt{5}}{2}$, $z_+ z_- = -1$, and $z_- < -1 - 0 - z_+ < 1$. Standard calculation allow you to find the coefficients A, B (which depends linearly on p and q) in

$$\begin{aligned} f(z) &= \frac{A}{z_+ - z} + \frac{B}{z_- - z} \\ &= \frac{A}{z_+} \sum_{n=0}^{\infty} \frac{z^n}{z_+^n} + \frac{B}{z_-} \sum_{n=0}^{\infty} \frac{z^n}{z_-^n} \\ &= A \sum_{n=0}^{\infty} \left(\frac{\sqrt{5}+1}{2} \right)^{n+1} + B \sum_{n=0}^{\infty} (-1)^n \left(\frac{\sqrt{5}-1}{2} \right)^{n+1}. \end{aligned}$$

Also, $r(f) = \min(|z_+|, |z_-|) = \frac{\sqrt{5}-1}{2}$ (the *golden ratio*!). For the coefficients,

$$a_n = a_n(f) = A \left(\frac{\sqrt{5}+1}{2} \right)^{n+1} + B \left(\frac{\sqrt{5}-1}{2} \right)^{n+1}.$$

If $A \neq 0$, the modulus of the first summand grows exponentially, while the other decreases exponentially. You are invited to work out A and B for the classical Fibonacci sequence (which grows exponentially!), and to find all cases when $A = 0$.

The procedure we have followed works as well if (2.43) is replaced by any other *linear, homogeneous, finite difference equations.*

2.3.4. *Trigonometry from within.* We have so far considered trigonometric functions as they arise from Euclidean geometry. As an innocuous intellectual exercise, we redefine them from scratch using power series, algebra, and a bit of calculus. This is not just for the sake of rigor: sometimes in maths excesses of rigor lead to *rigor mortis* and, besides, Euclidean geometry itself is historically the paradigm of rigor, although it fell out of fashion in contemporary education. A better motivation is that when a model of a theory is built on calculations, it is easier transplanting the theory, or parts of it, in different contexts. A contemporary, perhaps even more stringent motivation, is that with the development of automated proofs, or computer assisted proofs, the process of rigorous, standardized formalization of statements and proofs has a very practical use.

We *define* the entire functions:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},\ \cos z = \frac{e^{iz}+e^{-iz}}{2},\ \sin z = \frac{e^{iz}-e^{-iz}}{2i}, \tag{2.45}$$

all of which are real for z real. The main relations concerning the exponential are:

$$e^{z+w} = e^z e^w,\ e^0 = 1,\ \frac{de^z}{dz} = e^z, \tag{2.46}$$

which immediately follow from the definition. The basic algebra of the trigonometric functions follows by their definition:

$$\cos(z+w) = \cos z\cos w - \sin z\sin w,\ cos(-z) = \cos z,\ \cos'(z) = -\sin z,\ \cos(0) = 1,$$

$$\sin(z+w) = \sin z\cos w + \cos z\sin w,\ \sin(-z) = -\sin(z),\ \sin'(z) = \cos z,\ \sin(0) = 0,$$

(2.47)

$$1 = \cos^2 z + \sin^2 z.$$

The missing characters are π and periodicity. To recover them, we restrict to $z = x$ real, and observe that both cosine and sine are solutions of the second order differential equation

$$y'' + y = 0. \tag{2.48}$$

Since $\cos(0) = 1$, $\cos(x) > 0$ on an interval $I = [0, \pi/2)$, where $\pi/2$ is maximal. We show that $\pi < \infty$. If it were not, cos would be strictly concave (because $\cos''(x) = -\cos(x) < 0$), hence $\lim_{x\to+\infty}\cos(x) = -\infty$ (because $\cos'(x) = -\sin(x) < 0$, at least for small $x > 0$), contradicting $|\cos(x)| \leq 1$. We have then $\pi < \infty$, $\cos(\pi/2) = 0$, and, since $\sin' = \cos$, sin increases in I up to $\sin(\pi/2) = 1$.

The behavior of cos and sin on $[\pi/2, 0)$, $[\pi/2, \pi)$, and $[-\pi, \pi/2)$, is easily reconstructed by that on $[0, \pi/2)$, by the symmetries of cos and sin with respect to $x = 0$, and by the relations

$$\cos(x + \pi/2) = -\sin(x),\ \sin(x + \pi/2) = \sin(x).$$

We know now how $\cos, \sin$ increase and decrease on $[-\pi, \pi]$. Also observe that

$$\cos(\pi) = -1 = \cos(-\pi),\ \cos'(\pi) = \sin(\pi) = 0 = \sin(-\pi) = \cos'(-\pi).$$

That is, both $\varphi(x) = \cos(x)$ and $\varphi(x) = \cos(x - 2\pi)$ satisfy the ODE 2.48 with the same initial conditions at $x = \pi$, hence they coincide together with their derivatives:

$$\cos(x - 2\pi) = \cos(x),\ \text{and}\ \sin(x - 2\pi) = \sin(x).$$

Since $T = 2\pi$ is the smallest positive number such that $\cos(T) = \cos(0)$, it is the *period* of cos and sin.

All identities relating exp, cos and sin extend from real (or imaginary) to complex variable, because the real (or imaginary) line has accumulation points in the complex plane. For y real, for instance,

$$e^{iy-2\pi i} = \cos(y - 2\pi) + i\sin(y - 2\pi) = \cos(y) + i\sin(y) = e^{iy},$$

which shows that exp is $2\pi i$-periodic (and that it is not Ti periodic if $0 < T < 2\pi$).

To finish, we recover our first definition of the complex exponential:

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y),$$

straight by the homomorphism property of exp and by the definition of cos, sin.

EXERCISE 18. *Prove that the length of the unit circle* $\mathbb{T}$ *is* 2π*, and that the area of the unit disc* $\mathbb{D}$ *is* π*.*

2.4. Removing the C^1 assumption: Goursat lemma. In the subsections above, we used the fact that the holomorphic map f is C^1 in order to apply some foundational results from Advanced Calculus (Volterra-Poincaré, Green). The definition of holomorphic map, and the Cauchy-Riemann equations, however, hold under the assumption that f is merely differentiable. In order to extend the results to the weaker regularity, we will pass through a lemma in which the Cauchy theorem for particular curves is proved for differentiable, holomorphic maps. From this we will deduce that if f is holomorphic, then it is in fact C^1, and all proofs we have presented above hold for it.

Goursat lemma is based on a subtle and clever estimate. Basically, differentiability says that f has a first degree Taylor expansion near any point a: the holomorphic, 1^{st} order polynomial is C^∞; the remainder might a priori be not too regular, but it is small.

THEOREM 11 (Goursat lemma). *Let* f *be holomorphic in a domain* Ω*, and let* R *be a closed triangle in it. Then,*

$$\int_{\partial R} f(z)dz = 0. \tag{2.49}$$

PROOF. Let $I_0 = \int_{\partial R} f(z)dz$, and divide R as the union of four triangles with disjoint interiors R_1, R_2, R_3, R_4, having as vertices those of R and the midpoints of R's edges. Since $\int_{\partial R} f(z)dz = \sum_{j=1}^4 \int_{\partial R_j} f(z)dz$, there is $R_j = R^{(1)}$ such that

$$|I_1| = \left| \int_{\partial R^{(1)}} f(z)dz \right| \geq |I_0|/4.$$

Split then $R^{(1)}$ into four triangles, and repeat the procedure. We find $R \supset R^{(1)} \supset R^{(2)} \supset \dots$ such that

$$|I^{(n)}| \geq |I_0|/4^n.$$

Also, the diameters of the triangles decrease exponentially,

$$\mathrm{diam}(R^{(n)}) = \mathrm{diam}(R)/2^n.$$

Let $\{a\} = \bigcap_{n\geq 1} R^{(n)}$. Since f is holomorphic at a, $f(z) = f(a) + f'(a)(z-a) + e(z)$, where $|e(z)|/|z-a| = o(1) \to 0$ as $z \to a$. Using the Cauchy theorem for the C^1 map $z \mapsto z - a$, we have:

$$\begin{aligned}
|I_0|/4^n &\leq |I^{(n)}| = \left| \int_{\partial R^{(n)}} f(z)dz \right| \\
&= \left| \int_{\partial R^{(n)}} [f(a) + f'(a)(z-a) + e(z)]dz \right| \\
&= \left| \int_{\partial R^{(n)}} e(z)dz \right| \\
&\leq 4\mathrm{diam}(R^{(n)}) \cdot \mathrm{diam}(R^{(n)}) o(1) \\
&= 4\mathrm{diam}(R)^2/4^n o(1).
\end{aligned}$$

As $n \to \infty$, this inequality implies that $I_0 = 0$. □

The fact that the work of $f(z)dz$ along triangles is null is all we need to find a potential, and this brings us back to the C^1 case.

THEOREM 12 (Morera's theorem). *If f is continuous in a domain Ω and $\int_c f(z)dz = 0$ on all triangles c contained in Ω together with their interior, then f is holomorphic and C^1 (hence, it is C^∞).*

In particular, if f is holomorphic in Ω, then it is C^1.

PROOF. The conclusion is local, hence we can work in $\Omega = D(a,r)$ and without loss of generality $a = 0$. We first find an antiderivative for f. Define $F(z) = \int_{[0,z]} f(w)dw$. For η real (we can assume $\eta > 0$), using Goursat lemma in the first equality, we have that

$$\begin{aligned} \frac{F(z+\eta)-F(z)}{\eta} - f(z) &= \frac{1}{\eta}\int_{[z,z+\eta]} f(w)dw - f(z) \\ &= \frac{1}{\eta}\int_0^\eta [f(z+t)-f(z)]dt \\ &\to 0 \end{aligned}$$

as $\eta \to 0$ by the continuity of f. Thus, $F_x = f$. A similar calculation gives $\frac{1}{i}F_y = f$:

$$\begin{aligned} \frac{F(z+i\eta)-F(z)}{i\eta} - f(z) &= \frac{1}{i\eta}\int_{[z,z+i\eta]} f(w)dw - f(z) \\ &= \frac{1}{\eta}\int_0^\eta [f(z+it)-f(z)]dt. \end{aligned}$$

Thus, F satisfies the Cauchy-Riemann equations, $F_x = f = -iF_y$. Thus, F is C^1 and holomorphic, hence it is C^∞. Then, $f = F'$ is C^∞ as well. □

Note that all we used here was that f is continuous and that it satisfies the thesis of Goursat lemma.

All theorems we proved earlier (Cauchy theorem, Cauchy formula, expansion in power series...), then, hold for all holomorphic functions.

Morera's theorem is a precious tool when we need to show that holomorphicity is preserved under some limiting procedure: pushing an integral inside a limit is generally easier that pushing a derivative. We give right away a rather surprising application of this principle.

A notion of convergence which turns to be very useful in holomorphic theory is that of *uniform convergence on compacta*. A sequence $f_n : \Omega_n \to \mathbb{C}$ converges uniformly on compact sets to $f : \Omega \to \mathbb{C}$ (Ω_n, Ω open in $\mathbb{C}$) if, for all compact subsets K of Ω, $f_n \to f$ uniformly on K.

THEOREM 13 (Weierstrass convergence theorem). *If the functions f_n are holomorphic and converge to f as n tends to infinity, then f is holomorphic.*

PROOF. Let c be a curve which is contained in a closed disc $\mathrm{cl}(D(a,r))$ in Ω. By the local Cauchy theorem and the assumption on uniform convergence,

$$0 = \lim_{n\to\infty}\int_c f_n(z)dz = \int_c f(z)dz.$$

By Morera's theorem, f is holomorphic. □

What's so surprising about it is that in one real variable the result fails in the most miserable way. The uniform convergence of the functions' sequence by itself just ensures continuity, and no more: the limit function could be nowhere differentiable, the graph of a 1-dimensional Brownian motion, and other highly non-smooth fractals.

2.5. More properties of holomorphic functions. We have so far several characterizations of the fact that a function f on a domain Ω is holomorphic.

(i) f ha complex derivative at any point.
(ii) f is differentiable and it satisfies the Cauchy-Riemann equations.
(iii) f is continuous and for each disc D in Ω and any triangle c in D we have $\int_c f(z)dz = 0$.
(iv) For each disc $D(a, r)$ in Ω, f can be expanded in $D(a, r)$ as a complex power series centered at a.

We will next extract from the equivalence of these definitions some important, almost immediate consequences.

2.5.1. *Some consequences of the Cauchy-Riemann equations.* We first observe the curious fact that, if f satisfies the Cauchy-Riemann equations in a region Ω, then

$$\det(Jf(z)) = (\partial_x u)^2 + (\partial_x v)^2 = |f'(z)|^2. \tag{2.50}$$

By the formula for the change of variables in the integrals,

$$\text{Area}(f(A)) = \int_A |f'(z)|^2 dxdy, \tag{2.51}$$

where the expression on the right denotes Euclidean area, taking into account multiplicities.

Theorem 14 (Inverse mapping theorem). *Let $f : \Omega \to \mathbb{C}$ be holomorphic, and suppose $f'(a) \neq 0$ at some $a \in \Omega$. Then, there are open neighborhood $U(a) \ni a$ and $V(f(a))$ of $f(a)$ such that $f : U(a) \to V(f(a))$ is bi-holomorphic (holomorphic with holomorphic inverse). Moreover, $(f^{-1})'(f(z)) = 1/f'(z)$ for z in $U(a)$.*

Proof. We know f is C^1 and $Jf(a) \equiv f'(a)$ is non-singular, hence the usual inverse mapping theorem applies, giving a C^1 inverse with

$$J(f^{-1})(f(a)) = Jf(a)^{-1}.$$

By item (i) of exercise 7, $Jf(a)^{-1}$ still has the form corresponding to the Cauchy-Riemann equations, hence f^{-1} is holomorphic. □

Theorem 15 (Angle conformal vs. holomorphic). *Suppose $f : \Omega \to \mathbb{C}$ is C^1 and angle-conformal and orientation preserving at $a \in \Omega$, and that the Jacobian of f is non-singular at a. Then, $\bar{\partial} f(a) = 0$. In particular, if f is angle-conformal and orientation preserving at all points a of Ω, then it is holomorphic.*

Proof. Consider complex numbers v with $|v| = 1$ (think of them as elements of the unit circle in the tangent space to $\mathbb{C}$ at a). Unravelling definitions, f is angle-conformal and orientation preserving at a if and only if

$$\partial f(a)v + \bar{\partial} f(a)\bar{v} = \lambda(v)e^{is}v \tag{2.52}$$

for some fixed real s and with $\lambda(v) \geq 0$. We can rewrite (2.52) as

$$[\partial f(a) - \lambda(v)e^{is}]v^2 = -\bar{\partial} f(a). \tag{2.53}$$

Taking moduli of both sides, for all v we have:

$$\lambda(v)^2 - 2\mathrm{Re}(e^{-is}\partial f(a))\lambda(v) + |\partial f(a)|^2 = |\bar{\partial} f(a)|^2.$$

Since λ is continuous in v, this implies that $\lambda(v) = \lambda_0$ is constant. By (2.53), this can only hold if $\partial f(a) - \lambda_0 e^{is} = 0$, hence $\bar{\partial} f(a) = 0$. The last assertion of the theorem follows from the Cauchy-Riemann equations. □

We have another notion of conformality, which is completely metric.

THEOREM 16 (Metrically conformal vs. holomorphic). *Suppose $f : \Omega \to \mathbb{C}$ is C^1, metrically conformal and orientation preserving at $a \in \Omega$. Then, $\bar{\partial} f(a) = 0$. In particular, if f is metrically conformal and orientation preserving at all points a of Ω, then it is holomorphic.*

PROOF. It is an immediate consequence of lemma 1. □

We are now justified in giving the following definition.

DEFINITION 3. *A* **conformal map** *from Ω_1 to Ω_2 (open subsets of the plane) is a holomorphic bijection $f : \Omega_1 \to \Omega_2$.*

We will see below that injectivity (as well as angle conformality) forces $f'(z) \neq 0$ for all $z \in \Omega_1$.

What about the implicit function theorem from Advanced Calculus? Does it have something interesting to tell us?

THEOREM 17 (Implicit function theorem in two complex variables). *Let Ω be open in $\mathbb{C}^2$, and $F = F(z, w) : \Omega \to \mathbb{C}$ be a C^1 function which is holomorphic in z and w separately: $\bar{\partial}_z F(z, w) = \bar{\partial}_w F(z, w) = 0$. Suppose that $F(a, b) = 0$ and that $\partial_w F(a, b) \neq 0$. Then, the exists an open neighborhood $U(a)$ of a in $\mathbb{C}$, and a holomorphic map $\varphi : U(a) \to \mathbb{C}$, such that $\varphi(a) = b$, and for all z in $U(a)$:*

$$F(z, \varphi(z)) = 0 \text{ and } \varphi'(z) = -\partial_z F(z, \varphi(z))/\partial_w F(z, \varphi(z)).$$

Moreover, φ is unique in the following sense. There is $\epsilon > 0$ such that if $z \in U(a)$, $F(z, w) = 0$, and $|w - b| < \epsilon$, then $w = \varphi(z)$.

EXERCISE 19. *Use the classical implicit function theorem to prove theorem 17.*

The C^1 assumption might be dispensed with, since it is implied by holomorphicity in each variable separately. This is a course on holomorphic functions of one variable, and proving this fact would lead us too far. But see, e.g., Hartogs' Theorem: separate analyticity implies joint analyticity by Paul Garrett for a self-contained proof.

2.5.2. *Some consequences of the power series characterization.* The zeros of a function are isolated and they have finite order.

THEOREM 18 (Zeros are a discrete set). *Let f be holomorphic in a domain Ω, and suppose that $f(a) = 0$. Then, either f vanishes identically, or there is $r > 0$ such that $f(z) \neq 0$ when $0 < |z - a| < r$. In this second case there exist $m \geq 1$ integer, and g holomorphic in Ω, such that*

$$f(z) = (z - a)^m g(z),$$

and $g(a) \neq 0$. In particular, if f does not vanish identically, its zeros are a discrete subset of Ω.

The number m is the *order of zero* of f at a. We set $m = 0$ if $f(a) \neq 0$.

PROOF. We can assume $a = 0$. If $D(0, r) \subseteq \Omega$, then

$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Either all coefficients of the power series vanish, then $f = 0$ on $D(0, r)$, or there is a first $m \geq 1$ such that $a_m \neq 0$:

$$f(z) = z^m \sum_{n=m}^{\infty} a_n z^{n-m}.$$

This shows that $g(z) = \begin{cases} f(z)/z^n \text{ if } z \neq 0 \\ a_m \text{ if } z = 0 \end{cases}$ defines a function which is holomorphic in Ω, $g(0) \neq 0$, and satisfies $f(z) = z^n g(z)$. Let's go back to the earlier case, where all coefficients vanish, hence f vanishes on $D(0, r)$. Let $E \subseteq \Omega$ be the set of the limit points of zeros of f (each point in E is a zero by continuity of f). The set E is a priori closed in Ω, and we have just shown that it is open. Since it is not empty and Ω is connected, $E = \Omega$. □

The following reformulation of theorem 18 will be tacitly used many times.

COROLLARY 6. *If f is holomorphic in a domain Ω, and it is constant on a subset E with an accumulation point in Ω, then f is constant on Ω.*

After a holomorphic change of variables, holomorphic maps are locally powers.

THEOREM 19 (Local behavior of a holomorphic map). *Let f be holomorphic and non-constant in a domain Ω, and let a a point if Ω. Then there exist $m \geq 1$ integer, open neighborhoos $U(a)$ of a and $V(f(a))$ of $f(a)$, and a bi-holomorphism φ : $U(a) \to V(f(a))$ such that $\varphi(a) = 0$ and $f(z) - f(a) = \varphi(z)^m$ in $U(a)$. Moreover, if $U(a)$ is small enough there exist exactly m such holomorphic functions φ.*

PROOF. Let $m \geq 1$ be the order of zero of $f(z) - f(a)$ at $z = a$, so that by theorem 18 we can write $f(z) = f(a) + (z - a)^m g(z)$ with g holomorphic in a neighborhood $U(a)$ and with $g(a) \neq 0$. By restricting the neighborhood, we can also assume that $U(a)$ is a disc and that $g(z) \neq 0$ in $U(a)$. Then, we can write $g(z) = \psi(z)^m$ with ψ holomorphic in $U(a)$, and $\psi(a) \neq 0$. ((*) root of a function)

We can choose ψ in a set $\mathcal{F}$ of m functions. Set $\varphi(z) = (z - a)\psi(z)$.

There are at least m functions φ, one for each choice of ψ. Viceversa, for each φ satisfying the thesis, $\psi(z) = \frac{\varphi(z)}{z-a}$ (defined as $\psi(a) = \varphi'(a)$ at $z = a$) satisfies

$$\psi^m(z) = \frac{f(z) - f(a)}{(z - a)^m} = g(z),$$

hence $\psi \in \mathcal{F}$. This shows that φ has to be one of the m functions we had chosen earlier. □

COROLLARY 7. *If f is angle-conformal and holomorphic in a neighborhood of a, then $f'(a) \neq 0$.*

PROOF. After the change of variables in theorem 19, the statement is that $z \mapsto z^m$ is angle-conformal if and only if $m = 1$, which is easily checked. □

In fact we might say more. If $m \geq 1$ is the number in the statement of theorem 19, and v is a unit vector at a, identifies as a unit complex number, then a curve c making an angle of t with the half axis $a + [0, +\infty)$ at a, is mapped to a curve making an angle of $mt + \alpha$ with $f(a) + [0, +\infty)$ at $f(a)$ (with α independent of t).

EXERCISE 20. *Find the value of α in terms of the value of f and its derivatives at a.*

COROLLARY 8 (Open mapping theorem). *If f is a non-constant holomorphic map defined in a domain Ω and $A \subseteq \Omega$ is open, then $f(A)$ is open.*

PROOF. Fix a in A, and apply theorem 19 with A instead of Ω. We can restrict $V(f(a))$ to have the form $V(f(a)) = D(f(a), \delta)$, in which case $f(U(a)) = D(f(a), \delta^m)$. Thus $f(\Omega)$ is open. □

COROLLARY 9 (Maximum principle for holomorphic functions). *Let f be holomorphic in a domain Ω If $|f|$ ha a local maximum, then f is constant.*

PROOF. Let a be a point in Ω and consider $D(a, r) \subset \Omega$. Since $f(a)$ lies in the interior of $f(D(a, r))$, $|f|$ does not have maximum modulus there. □

2.5.3. *Cauchy estimates and Liouville I.* When we expand a function f holomorphic in Ω as a power series in a disc $D(a, r)$ having closure contained in Ω,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \tag{2.54}$$

the coefficients of the series are given by

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z-a}^{n+1} dz.$$

An estimate on the size of the coefficients in terms of the size of f immediately follows. Let

$$M_{f,a}(r) = \max\{|f(z)| : |z-a| = r\}. \tag{2.55}$$

THEOREM 20 (Cauchy estimates). *If f is holomorphic in a domain containing the closure of $D(a, r)$ has the series expansion (2.54), then*

$$|a_n| \leq \frac{M_{f,a}(r)}{r^n}. \tag{2.56}$$

A holomorphic function on the whole complex plane is called *entire.*

THEOREM 21 (Liouville theorem). *Let f be entire. If f is bounded, then f is constant.*

PROOF. Let C be an upper bound for $|f(z)|$. For $n \geq 1$, by (2.56) we have that, for all $r > 0$,

$$|a_n| \leq \frac{C}{r^n} \to 0$$

as $r \to \infty$. □

THEOREM 22 (Fundamental theorem of algebra). *If p is a nonconstant polynomial, then $p(z) = 0$ has at least a solution.*

From algebra, we know that, then, $p(z) = 0$ has $\deg(p)$ solutions, taking multiplicities into account.

PROOF. It is easily verified that $\lim_{z\to\infty} p(z) = \infty$. If $p(z) = 0$ had no solutions, $\frac{1}{p(z)}$ would be a bounded, entire function, so it would be constant by Liouville theorem. □

2.5.4. *Some consequences of the complex integral approach.* Using together line integrals and series, we can prove an important result by Hurwitz in its basic form.

THEOREM 23 (Basic Hurwitz theorem). *Let $\{f_n\}$ be a sequence of holomorphic functions on a domain Ω, which converges uniformly on compact sets to f. If the functions f_n have no zero in Ω, then, either f has no zero in Ω, or f vanishes identically on Ω.*

PROOF. Suppose a is an isolated zero of f, $f(z) = (z-a)^m g(z)$ with $m \geq 1$ and $g(a) \neq 0$. Let $D(a,r)$ be a disc with closure contained in Ω such that g (hence g) has no zeros for $|z-a| = r$. By the Cauchy estimates, we have that $f_n \to f$ and $f_n' \to f'$ uniformly of $\partial D(a,r)$, hence,

$$\begin{aligned} \lim_{n\to\infty} \frac{f_n'(z)}{f_n(z)} &= \frac{f'(z)}{f(z)} \\ &= \frac{m}{z-a} + \frac{g'(z)}{g(z)}. \end{aligned}$$

Integrating, passing the limit inside the integral (which we can do because the convergence is uniform), and using the fact that $\frac{g'}{g}$ is holomorphic in cl D(a,r),

$$\begin{aligned} m &= \int_{\partial D(a,r)} \left(\frac{m}{z-a} + \frac{g'(z)}{g(z)} \right) dz \\ &= \lim_{n\to\infty} \int_{\partial D(a,r)} \frac{f_n'(z)}{f_n(z)} dz \\ &= 0, \end{aligned}$$

and we have a contradiction. If there are no isolated zeros, either f vanishes identically, or it is zero-free. □

2.6. Classification of singularities.

2.6.1. *Isolated singularities.*

THEOREM 24 (Theorem on removable singularities). *Let f be holomorphic in $\Omega \setminus \{a\}$, where $a \in \Omega$, a region in the plane. If*

$$\lim_{z\to a} (z-a) f(z) = 0, \tag{2.57}$$

then $f(a) := \lim_{z\to a} f(z)$ exists in $\mathbb{C}$, and f is so extended to a function which is holomorphic on Ω.

This theorem is an avatar of a general principle: the solution to certain differential equations can not have a "substantial" isolated singularity where the function grows below a certain rate. The minimal rate of growth which is allowed is often that of the "fundamental solution" to the equation.

PROOF. Define

$$g(z) = \begin{cases} (z-a)f(z) \text{ if } z \neq a, \\ 0 \text{ if } z = a. \end{cases}$$

Then, g is continuous in Ω, holomorphic in $\Omega \setminus \{a\}$, and this all we need in order to show that

$$\int_c g(z)dz = 0 \tag{2.58}$$

for all triangles c contained in a disc centered at a and contained in Ω. In fact, (2.58) holds if a is a vertex of c, since we can remove from the "solid" a small triangle t_ϵ with vertex at a and diameter less than ϵ. If c_ϵ is the quadrangle containing what is left,

$$\int_c g(z)dz = \int_{c_\epsilon} g(z)dz + \int_{t_\epsilon} g(z)dz = \int_{t_\epsilon} g(z)dz \to 0$$

as $\epsilon \to 0$, by the continuity of g. If a is on an edge, or in the interior, we split the triangle in two, or three, triangles with vertex at a.

We can then apply Morera's theorem, showing that g is holomorphic in Ω, then C^∞, hence g' is holomorphic in Ω as well. But $g' = f$ on $\Omega \setminus \{a\}$, and we are done. □

When f is holomorphic in $\Omega \setminus \{a\}$, where $a \in \Omega$, a region in the plane, we say that f has an *isolated singularity* at a.

Next we have a quantization of the isolated singularities of blow-up type.

THEOREM 25. *Suppose f has an isolated singularity at $a \in \Omega$, and that*

$$\lim_{z \to a} f(z) = \infty.$$

Then, there exists $n \geq 1$, and g holomorphic in Ω, $g(a) \neq 0$, such that

$$f(z) = \frac{g(z)}{(z-a)^n}. \tag{2.59}$$

In this case we say that f has a *pole* of *order* n at a.

PROOF. The function $1/f(z)$, defined in a neighborhood of $z = 0$, tends to 0 as $z \to a$, hence $z = 0$ is a removable singularity for $1/f$ and for some $n \geq 1$,

$$\frac{1}{f(z)} = (z-a)^n g(z)$$

in a neighborhood $U \setminus \{a\}$ of a (with a removed), where φ is holomorphic in U and $g(a) \neq 0$. This implies that $f(z)(z-a)^n = g(z)$ is holomorphic in U, and it is then holomorphic in the whole of Ω. □

When f has an isolated singularity at a which is neither a pole, nor removable, we say that the singularity is *essential*. Example of essential singularities are easy to produce. For instance,

$$f(z) = e^{1/z}$$

has an essential singularity at $z = 0$: $\lim_{x \to 0_+} xf(x) = +\infty$, which rules out a removable singularity; $\lim_{x \to 0_-} f(x) = 0$, which rules out a pole.

The behavior of a holomorphic f near an essential singularity is rather chaotic. This is best expressed by Picard's Little and Big theorems, which we postpone to a later section. A weaker, by meaningful statement is the following.

THEOREM 26 (Casorati-Weierstrass theorem). *Suppose f is holomorphic in $\Omega \setminus \{a\}$ and it has an essential singularity at a. Then, $f(\Omega)$ is dense in $\mathbb{C}$.*

An interesting feature of the statement is that we can shrink Ω at will around a, and obtain the same conclusion.

COROLLARY 10. *If a is an essential singularity for f, then the limit set of $f(z)$ as $z \to a$ is the Riemann sphere $\widehat{\mathbb{C}}$.*

PROOF. Suppose $b \in \mathbb{C}$ is not a limit point of f as $z \to a$. We can assume $b \neq \infty$ because the set of the limit points is closed in the Riemann sphere. Then, after possibly shrinking Ω, $h(z) = \frac{1}{f(z)-b}$ is holomorphic in $\Omega \setminus \{a\}$, and bounded. Hence, $z = a$ is a removable singularity for h, and $f(z) = b + \frac{1}{h(z)}$. Either $h(a) \neq 0$, hence is holomorphic in a neighborhood of a, or $h(a) = 0$, and a is a pole for f. In both cases, a is not an essential singularity. □

2.6.2. *Liouville II and III.* With the preceeding results at haned, we can show a sharper version of the Liouville property and the Liouville property for harmonic functions.

THEOREM 27 (Sharper Liouville theorem). *Let f be an entire function. If f omits an open set, then it is a constant.*

PROOF. If $\lim_{z\to\infty} f(z) = \infty$, then the proof of the fundamental theorem of algebra implies that the equation $f(z) = c$ has always a solution, unless f is constant, hence f does not omit any value.

Otherwise, consider $g(z) = f(1/z)$. If it is bounded in a neighborhood of zero, then it can be redefined there to become a bounded entire function, which is constant. The case of a pole was considered above, and the last possibility is that $z = 0$ is an essential singularity. By Casorati-Weierstrass, then, g, hence f, has dense range in $\mathbb{C}$. □

THEOREM 28 (Liouville property for harmonic functions). *Let $u : \mathbb{C} \to \mathbb{R}$ be a positive harmonic function (we could as well assume it's bounded from below, or from above). Then, u is constant.*

PROOF. Since $\mathbb{C}$ satisfies the Volterra-Poincaré property, there is a harmonic v such that $u + iv = f$ is entire. Since $u > 0$, f omits the whole right-half plane, hence, by 27, it has to be constant. □

These theorems could also be derived in one stroke from the Riemann mapping theorem, which is however a much deeper result than the ones we have seen up to this point.

3. The Riemann sphere and the hyperbolic disc

In this section we consider holomorphic functions $f : \Omega \to \Omega$ when $\Omega = \mathbb{C}_*$ (the one-point compactification of the complex plane), and $\Omega = \mathbb{D}$ (the unit disc). In both cases, we characterize the full *automorphism group*, the class of those f which are bijective. These considerations have a strong geometric content, which was already clear to Riemann, and have to do with the geometry of the sphere (in the case of $\mathbb{C}_*$), and that of the *non-Euclidean plane*.

The case of the unit disc is especially interesting, since the holomorphic maps of the disc into itself are many. The main basic result, which can also rephrased (and we will) in geometric terms, is Schwarz' lemma.

3.1. The Riemann sphere.

3.1.1. *On polynomials and rational functions.* We can extend the notion of singularity at ∞. If $f : \Omega \to \mathbb{C}$ be defined on $\Omega \setminus \{\infty\}$, where $\Omega \subset \mathbb{C}_*$ is open, then *the type of singularity of f at ∞* is, by definition, the same that $f(1/z)$ has at 0. For instance, a polynomial of degree d has a pole of degree d at ∞. The converse holds.

Theorem 29. *Let $f : \mathbb{C} \to \mathbb{C}$ be entire (holomorphic on the whole plane). f has a pole at ∞, then f is a polynomial.*

Proof. If $f(1/z)$ has a pole of degree d at 0, then $z^d f(1/z) = g(z)$ is defined on $\mathbb{C}$, has a removable singularity at 0, and it can be extended to be holomorphic on $\mathbb{C}$, with $g(0) \neq 0$. That is, $f(z) = g(1/z)z^d$ for $z \neq 0$, and $g(0) \neq 0$. The Cauchy estimates fot the n^{th} Taylor coefficient a_n of f give

$$|a_n| \leq \frac{M_{f,0}(r)}{r^n} \leq C\frac{r^d}{r^n}$$

if d is the order of the pole at ∞, hence $a_n = 0$ for $n > d$. □

Theorem 30 (Characterization of the conformal maps of the plane). *The entire, conformal (surjective) maps $f : \mathbb{C} \to \mathbb{C}$ are the non-degenerate linear maps, $f(z) = az + b$.*

The hypothesis that f be surjective can be dropped, but in order to do that we need the sharp version of the Volterra-Poincaré theorem or, even better, the Riemann mapping theorem.

Proof. If ∞ were an essential singularity for f, by the Casorati-Weierstrass theorem the image of $\{z : |z| > 1\}$ would be dense in $\mathbb{C}$, and so, by the open mapping theorem, $f(\{z : |z| > 1\}) \cap f(\{z : |z| < 1\}) \neq \emptyset$, which kills injectivity. Then ∞ is a pole and, by theorem 29, f is a polynomial of degree $d \geq 1$. If $d \geq 2$, then $f'(z) = 0$ has a solution by the fundamental theorem of algebra, hence f can not be injective. Thus, $f(z) = az + b$ with $a \neq 0$. □

The *conformal group* of the plane, then, consists in the non-degenerate linear maps, with four real degrees of freedom. We know from Euclidean geometry that the isometries correspond to $|a| = 1$ (three degrees of freedom), and that the extra parameter corresponds to to homotheties.

Lemma 2. *Let $f : \mathbb{C}_* \to \mathbb{C}_*$ be holomorphic and non-constant. Then, it is surjective.*

Proof. By the open mapping theorem, $f(\mathbb{C}_*)$ is open in $\mathbb{C}_*$; but $\mathbb{C}_*$ is also compact, hence $f(\mathbb{C}_*)$ is also closed. Since $\mathbb{C}_*$ is connected, $f(\mathbb{C}_*) = \mathbb{C}_*$. □

Corollary 11 (the conformal group of the sphere). *The by-holomorphic maps $f : \mathbb{C}_* \to \mathbb{C}_*$ are the fractional linear transformations.*

Exercise 21. *Use lemma 2 and theorem 30 to prove corollary 11.*

The conformal group of the extended plane, then, has dimension six.

Theorem 31 (Holomorphic functions on the sphere). *The holomorphic maps $f : \mathbb{C}_* \to \mathbb{C}_*$ are the rational maps.*

PROOF. We can suppose that f is non-constant. By lemma 2, f is surjective. After composing with with a fractional linear transformation, we can suppose that $f(\infty) = 0$. There are just finitely many $z_1, \dots, z_n$ such that f has a pole of order m_j at z_j: is there were infinitely many, they would accumulate at some $a \in \mathbb{C}$ After composition with another fractional linear transformations, we would have a function whose zeros have an accumulation point, then it would be constant. The function

$$g(z) = f(z)(z - z_1)^{m_1} \dots (z - z_n)^{m_n}$$

is then entire and it extends to a function which is holomorphic on $\mathbb{C}_*$, which is then a polynomial by theorem 29. Hence, f is a rational function. □

THEOREM 32 (Zeros and poles of a rational function). *Let $f : \mathbb{C}_* \to \mathbb{C}_*$ be holomorphic (i.e. rational) and non-constant. Then, f has the same number of poles and zeros (counting multiplicities).*

The number of zeros/poles is called the *degree* of f, $\deg(f) = \max\{\deg(p), \deg(q)\}$.

PROOF. Write $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials with no common factor. The difference between number of poles and zeros in $\mathbb{C}$ is $\deg(q) - \deg(p)$. On the other hand, if $\deg(q) > \deg(p)$, then f has a zero of order $\deg(q) - \deg(p)$ at ∞, and the total balance is then $\deg(q) - [\deg(p) + (\deg(q) - \deg(p))] = 0$. Same reasoning when $\deg(q) < \deg(p)$ or $\deg(q) = \deg(p)$. □

The class of the holomorphic maps from $\mathbb{C}_*$ to itself, as in the more general case of the *compact Riemann surfaces* is not very large, and its study has more an algebraic, geometric, and combinatorial flavor, rather than analytic. We mention here some covering properties of of the rational functions.

Let $r(z)$ be a rational function of degree d. By theorem 32, for each value c in $\mathbb{C}_*$, the equation $r(z) = c$ has d solutions, taking multiplicities into account. Consider the *critical values* c of r, those for which $\sharp(r^{-1}(c)) < d$. After a change of coordinates, we can assume that $c \in \mathbb{C}$ and that $r^{-1}(c) \subset \mathbb{C}$. There is at least a *critical point* $a \in r^{-1}(c)$ such that $r(z) - c = k(z - a)^m + O((z - a)^{m+1})$ with $m \geq 2$, hence, $r'(a) = 0$. Unless r is constant, the zero set of r' is discrete in $\mathbb{C}_*$, hence it is finite. Thus,

(i) there are finitely many *critical values* c in $\mathbb{C}_*$ such that $\sharp(r^{-1}(c)) < d$;
(ii) for any other point $c \in \mathbb{C}_*$, $\sharp(r^{-1}(c)) = \{a_1, \dots, a_d\}$ and there is a neighborhood V_c of c and disjoint neighborhoods U_{a_j} of the a_j's such that $r^{-1}(V_c) = U_{a_1} \cup \dots \cup U_{a_d}$; moreover,
(iii) $r : U_{a_j} \to V_c$ is a bi-holomorphism.

Let C be set of the critical points. Items (ii-iii) say that $r : \mathbb{C}_* \setminus r^{-1}(C) \to \mathbb{C}_* \setminus C$ is a *(holomorphic) covering map*. Together with (i), $r : \mathbb{C}_* \to \mathbb{C}_*$ is a *branching covering map*.

3.1.2. *The projective line.* The *complex projective line* $\mathbb{C}P^1$ is populated by the equivalence classes $[z; \zeta]$ in which $\mathbb{C}^2 \setminus \{(0,0)\}$ is partitioned by the equivalence relation

$$(z, \zeta) \sim (z', \zeta') \iff \exists \lambda \in \mathbb{C} \setminus \{0\} : (z', \zeta') = \lambda(z, \zeta). \tag{3.1}$$

The line $\mathbb{C}P^1$ is endowed with the quotient topology coming from that of $\mathbb{C}^2 \setminus \{(0,0)\}$. We identify $\mathbb{C}P^1$ with $\mathbb{C}_*$ via the map $\Theta : \mathbb{C}P^1 \to \mathbb{C}_*$:

$$\Theta([z; 1]) = z, \ \Theta([1; 0]) = \infty. \tag{3.2}$$

EXERCISE 22. *Show that* Θ *is a homeomorphism.*

What is more relevant to us, is that fractional transformations have a very simple interpretations in terms of the projective line.

PROPOSITION 7. (i) *Let* $A \in SL(2,\mathbb{C})$. *If* $\begin{pmatrix} z \\ \zeta \end{pmatrix} \sim \begin{pmatrix} z' \\ \zeta' \end{pmatrix}$, *then* $A\begin{pmatrix} z \\ \zeta \end{pmatrix} \sim A\begin{pmatrix} z' \\ \zeta' \end{pmatrix}$. *That is,* A *induces a map* $\Phi_A : \mathbb{C}P^1 \to \mathbb{C}P^1$.

(ii) $\Phi_A = \Phi_B$ *if and only if there is* $\lambda \in \mathbb{C} \setminus \{0\}$*:* $B = \lambda A$.

(iii) *Let* $\Psi_A(z) = \frac{az+b}{cz+d}$ *if* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then,* $\Theta \circ \Phi_A = \Psi_A \circ \Theta$. *That is, we have an isomorphic identification of the* **projective transformations** Ψ_A *on* $\mathbb{C}P^1$ *and the fractional linear maps on* $\mathbb{C}_*$.

These statements are harder to read than to prove, and details are left to the reader. We will not develop this viewpoint, which is nonetheless important in algebraic geometry, Lie group theory, and several complex variables.

We return instead to (2.26), which provided a fractional linear transformation mapping $z_j \mapsto j$ for $j = 0, 1, \infty$. More generally, the *cross-ratio* of four points, of which at least three are distinct, $z_1, z_2, z_3, z_4 \in \mathbb{C}_*$ (in the order) is

$$(z_1 z_2 z_3 z_4) = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} \in \mathbb{C}_*. \tag{3.3}$$

EXERCISE 23. *Show how* $(z_1 z_2 z_3 z_4)$ *changes under permutations of its arguments.*

How does it look in projective coordinates $[z;\zeta]$? A quick (and 99% rigorous) calculation gives (with $z \equiv [w;\zeta]$, so to speak):

$$\begin{aligned}
(z_1 z_2 z_3 z_4) &= \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} \\
&= \left[\frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4}; 1\right] \\
&= \left[\frac{z_1 - z_3}{z_1 - z_4}; \frac{z_2 - z_3}{z_2 - z_4}\right] \\
&= [(z_1 - z_3)(z_2 - z_4); (z_1 - z_4)(z_2 - z_3)] \\
&= [(w_1\zeta_3 - w_3\zeta_1)(w_2\zeta_4 - w_4\zeta_2); (w_1\zeta_4 - w_4\zeta_1)(w_2\zeta_3 - w_3\zeta_2)] \\
&= \left[\det\begin{pmatrix} w_1 & w_3 \\ \zeta_1 & \zeta_3 \end{pmatrix} \det\begin{pmatrix} w_2 & w_4 \\ \zeta_2 & \zeta_4 \end{pmatrix}; \det\begin{pmatrix} w_1 & w_4 \\ \zeta_1 & \zeta_4 \end{pmatrix} \det\begin{pmatrix} w_2 & w_3 \\ \zeta_2 & \zeta_3 \end{pmatrix}\right] \\
&= \dots
\end{aligned}$$

The last two lines make fully sense in $\mathbb{C}P^1$.

PROPOSITION 8. (i) *Projective transformations preserve the cross-ratio.*

(ii) *Any map preserving the cross-ratio is a projective transformation.*

PROOF. (i) We can verify this in $\mathbb{C}_*$. There, a few strokes show that the cross-ratio is preserved under translation, multiplication times a constant, and $z \mapsto 1/z$.

(ii) Suppose f preserves the cross-ratio, i.e., for distinct, fixed points z_1, z_2, z_3, and $z \in \mathbb{C}_*$,

$$(f(z), f(z_1), f(z_2), f(z_3)) = (z, z_1, z_2, z_3).$$

From the equation, we can compute $f(z)$, which turns out to be a fractional linear transformation in the variable z. $\square$

The cross-ratio also characterizes lines and circles in the extended plane.

THEOREM 33 (cross-ratio, and circles and straight lines). *Let z_1, z_2, z_3, z_4 be distinct points in $\mathbb{C}_*$. Then, they lie on a line or a circle if and only if $(z_1 z_2 z_3 z_4)$ is real.*

PROOF. After composing with a fractional linear transformation, we can suppose that the points are all in the finite plane. We have that (modulo 2π)

$$\arg(z_1 z_2 z_3 z_4) = \arg \frac{z_1 - z_3}{z_1 - z_4} - \arg \frac{z_2 - z_3}{z_2 - z_4}.$$

Euclidean geometry of the circle shows that the difference is either 0 or π, and that they are not on the same circle for different values. $\square$

3.1.3. *The Riemann sphere.* A more quantitative way to look at the extended plane $\mathbb{C}_*$ is by means of stereographic projections. Let $S^2 = \{(w,t) \in \mathbb{C} \times \mathbb{R} : |w|^2 + t^2 = 1\}$ be the unit sphere in three-dimensional Euclidean space, and define the (anti)-projection $P_+ : \mathbb{C} \to S^2 \setminus \{(0,1)\}$,

$$P_+(z) = \left(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2-1} \right). \tag{3.4}$$

Geometrically, $P_+(z)$ is the unique intersection of the straight line through $(z,0)$ and the North Pole $(0,1)$ with S^2.

EXERCISE 24. *Verify the last assertion.*

The second (anti)-projection maps $\mathbb{C}$ onto $S^2 \setminus \{(0,-1)\}$,

$$P_-(z) = P_+(1/z) = \left(\frac{2\bar{z}}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2} \right). \tag{3.5}$$

The maps P_+^{-1} and P_-^{-1} are charts making S^2 into a 2-dimensional manifold according to the usual definition. Since the transition map

$$P_-^{-1} \circ P_+(z) = 1/z$$

is holomorphic on $\mathbb{C} \setminus \{0\}$, the following definition of holomorphic function is coherent. A function $F : S^2 \to S^2$ is *holomorphic* if the maps $P_\sigma^{-1} \circ F \circ P_\tau$ are holomorphic on their domain for $\sigma, \tau \in \{\pm\}$.

But let's first pull the spherical geometry back to $\mathbb{C}_*$ by means of P_+.

PROPOSITION 9. *Let*

$$ds^2 = |P_+(z+dz) - P_+(z)|^2 \tag{3.6}$$

be the pull-back of the spherical metric on $\mathbb{C}$ by P_+. Then,

$$ds^2 = \frac{4|dz|^2}{(1+|z|^2)^2}. \tag{3.7}$$

PROOF.

$$|P_+(z+dz) - P_+(z)|^2 = \left| \frac{2(z+dz)}{|z+dz|^2+1} - \frac{2z}{|z|^2+1} \right|^2 + \left| \frac{|z+dz|^2-1}{|z+dz|^2+1} - \frac{|z|^2-1}{|z|^2+1} \right|^2$$

$$\begin{aligned}
&= 4\left|\frac{(|z|^2+1)dz - z(\bar{z}dz + zd\bar{z})}{(|z|^2+1)^2}\right|^2 + 4\left|\frac{(\bar{z}dz + zd\bar{z})}{(|z|^2+1)^2}\right|^2 \\
&= \frac{4}{(|z|^2+1)^4}\left[|dz - z^2 d\bar{z}|^2 + |\bar{z}dz + zd\bar{z}|^2\right] \\
&= \frac{4|dz|^2}{(|z|^2+1)^4}[1 + |z|^4 + 2|z|^2] \\
&= \frac{4|dz|^2}{(1+|z|^2)^2}.
\end{aligned}$$

□

The fact that points which are far away on the plane can be very close on the sphere is encoded in the factor $\frac{2}{1+|z|^2}$. Think of a small circle on the sphere, centered at the North Pole: it is mapped to a circle with a very large radius on the complex plane.

COROLLARY 12. *P_+ and P_- are conformal maps.*

PROOF. The metric in (3.7) is conformal to the Euclidean metric: angles are computed as in the Euclidean plane, and the infinitesimal circle in the plane having center z and radius $|dz|$ is mapped by P_+ onto an infinitesimal circle on S^2 having center $P_+(z)$ and radius $\frac{2|dz|}{1+|z|^2}$. □

EXERCISE 25 (A small lab on spherical geometry). *Since our early experiences with globes we learn that: (a) the geodesics on the sphere are maximal circles (such as the "meridians"); (b) the other circles (such as the "parallels") are the curves at fixed spherical distance from a geodesic; (c) both can be viewd as metric circles in the spherical geometry.*

(i) *Show that the stereographic projection of the circles on S^2 are the circles and the straight lines in $\mathbb{C}$. (You might do it without calculations, assembling facts we have already proved).*

(ii) *Show that the straight lines and circles corresponding to spherical geodesics are those passing through two diametrically opposed points in the circle $S^2 \cap \mathbb{C} \times \{0\}$.*

(iii) *Show that a fractional linear map is an isometry in the spherical metric if and only if there is a geodesic which is sent to a geodesic.*

(iv) *Let $f : \mathbb{C}_* \to \mathbb{C}_*$ be holomorphic (hence, rational), and let d_s be the spherical metric. Find the explicit formula for the distorsion of d_s by f,*

$$\delta_{s,s} f(z) = \frac{d_s(f(z+dz), f(z))}{d_s(z+dz, z)}.$$

Unless f is constant, f can not be a strict contraction (i.e.: with Lipschitz constant less than 1) for the spherical metric. Why?

(v) *Use (iii), or (iv) and direct calculation, to characterize the spherical isometries among fractional linear maps (you should get a family depending on three real parameters). How would you interpret the other fractional linear transformations?*

3.2. Schwarz lemma and the hyperbolic plane. In the first subsection we state and prove Schwarz lemma and some of its extensions. The lemma itself is the main link between hyperbolic geometry in the plane and holomorphic function theory. Some of the connections, together with a not too rigorous introduction to hyperbolic geometry, are discussed in the following two subsections. They can be skipped without affecting much of the content of the other chapters, but the hyperbolic viewpoint is central to much contemporary holomorphic theory, and I think that a crash introduction to it is worth the time.

3.2.1. *Schwarz lemma and Schwarz-Pick lemma.* For the unit disc in the complex plane we use the common notation $\mathbb{D} = D(0,1)$. Schwarz' lemma imposes an important constraint on holomorphic maps from $\mathbb{D}$ into itself.

THEOREM 34 (Schwarz's lemma). *Let f be holomorphic from $\mathbb{D}$ into itself, $f(0) = 0$. Then,*

(i) $|f(z)| \leq |z|$, *and equality holds for some $z \neq 0$ if and only if $f(z) = e^{is}z$ is a rotation;*
(ii) $|f'(0)| \leq 1$, *and $|f'(0)| = 1$ if and only if f is a rotation.*

PROOF. Define $g : \mathbb{D} \to \mathbb{C}$, $\begin{cases} g(z) = \frac{f(z)}{z} \text{ if } 0 < |z| \leq r, \\ f'(0) \text{ if } z = 0. \end{cases}$, which is holomorphic. For fixed $0 < r < 1$, consider the restriction of g to $\mathrm{cl}(D(0,r))$. Since g is holomorphic in $D(0,r)$ and continuous on $\mathrm{cl}(D(0,r)$, its maximum is achieved at the boundary $\partial D(0,r)$ by the maximum principle, $|g(z)| \leq \max\{|g(e^{it})| : t \in \mathbb{R}\} \leq 1/r$. This holds for all $r < 1$, so $|g(z) \leq 1|$. Hence,

$$|f(z)| \leq |z|$$

and $|f'(0)| \leq 1$; which shows the inequalities in (i) and (ii). If we had equality in (i) for some $z \neq 0$, or in (ii), g would have an interior maximum point, hence it would be constant: $g(z) = c$ with $|c| = 1$. This gives $f(z) = e^{is}z$ for some s. □

COROLLARY 13. *If φ is a bi-holomorphic map of $\mathbb{D}$ onto itself, and $\varphi(0) = 0$, then $\varphi(z) = e^{is}z$ is a rotation around the origin.*

PROOF. We have $1 = |(\varphi \circ \varphi^{-1})'(0)| = |\varphi'(0)| \cdot |(\varphi^{-1})'(0)|$, and by (ii) in theorem 34, we have that both $|\varphi'(0)| = |(\varphi^{-1})'(0)| = 1$, hence φ is a rotation. □

Next, we use Schwarz lemma to characterize all bi-holomorphic maps of $\mathbb{D}$ onto itself.

THEOREM 35 (the conformal group of the disc). *The bi-holomorphims $\varphi : \mathbb{D} \to \mathbb{D}$ are exactly the maps having the form*

$$\varphi(z) = e^{is}\frac{a - z}{1 - \bar{a}z}, \tag{3.8}$$

with s real and $|a| < 1$.

Such linear transformations φ are also called *automorphisms* of $\mathbb{D}$, and their family is called the *Möbius group* of $\mathbb{D}$, $M(\mathbb{D})$. It is the subgroup of the group of the linear transformations which fixes the unit disc.

PROOF. We first show that the maps (3.8) map the disc onto itself. We can assume $s = 0$. For z in the disc,

$$\begin{aligned} 1 - |\varphi(z)|^2 &= \frac{|1 - \bar{a}z|^2 - |a - z|^2}{|1 - \bar{a}z|^2} \\ &= \frac{1 + |z|^2|a|^2 - |a|^2 - |z|^2}{|1 - \bar{a}z|^2} \end{aligned}$$

$$\begin{aligned} &= \frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|^2} \\ &\geq 0, \end{aligned} \tag{3.9}$$

hence φ injectively maps $\mathbb{D}$ into itself (and the circle to the circle). We have to show that φ is a bijection. We could use a result from topology, or observe that $w = \varphi(z)$ is the inverse of itself:

$$\bar{a}zw - z - w + 1 = 0,$$

which is symmetric in z and w.

Next, we use the easy part of the theorem and the corollary to Schwarz lemma to prove the more surprising part. Suppose φ is a bi-holomorphism, and suppose that $\varphi(0) = a$. If $\psi(z) = \frac{a-z}{1-\bar{a}z}$, then $\psi \circ \varphi$ is a bi-holomorphism of $\mathbb{D}$ which fixes the origin, hence $\psi(\varphi(z)) = e^{is}z$. Since $\psi = \psi^{-1}$,

$$\varphi(z) = \psi(e^{is}z) = \frac{a - e^{is}z}{1 - \bar{a}e^{is}z} = e^{is}\frac{ae^{-is} - z}{1 - \overline{ae^{-is}}z},$$

which belongs to the Möbius group. □

We have met the "magic relation" (3.9) satisfied by the autmorphisms of the disc, and we add here a second one concerning the automorphisms' derivatives,

$$\frac{1}{1-|z|^2} = \frac{|\varphi'(z)|}{1-|\varphi(z)|^2}. \tag{3.10}$$

EXERCISE 26. *Verify* (3.10).

Observe that the automorphism group is a Lie group having (real) dimension 3.

After reading the statement of Schwarz' lemma, you have probably wondered what happens if the assumption $f(0) = 0$ is removed. Having Möbius maps to move points around, we can easily answer.

THEOREM 36 (Schwarz-Pick lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then,*

(i) *For $z, w \in \mathbb{D}$ we have:*

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)} \right| \leq \left| \frac{w - z}{1 - \bar{w}z} \right|, \tag{3.11}$$

and equality holds for a couple $z \neq w$ if and only if f is an automorphism (in which case (3.11) *is an identity);*

(ii) *For $z \in \mathbb{D}$ we have:*

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1-|z|^2}, \tag{3.12}$$

with equality for some z if and only if f is an automorphism (in which case (3.12) *is an identity).*

We could formally deduce (3.12) from (3.11) by considering $w = z + dz$: it's not very rigorous, but I find it convincing enough. Unfortunately, this way we do not obtain the characterization of the cases when we have equality. The inequalities in the theorem correspond to the "magic identities" satisfied by the automorphisms.

PROOF. We move both w and and $f(w)$ by means of the maps

$$\varphi(\xi) = \frac{f(w) - \xi}{1 - \overline{f(w)}\xi}, \ \psi(\zeta) = \frac{w - \zeta}{1 - \bar{w}\zeta},$$

so that $\varphi^{-1} = \varphi$, $\psi^{-1} = \psi$, and $\varphi \circ f \circ \psi$ maps $0 \mapsto w \mapsto f(w) \mapsto 0$. We then apply part (ii) of Schwarz' lemma and (3.10) to obtain:

$$\begin{aligned} 1 &\geq |(\varphi \circ f \circ \psi)'(0)| = |\varphi'(f(w))| \cdot |f'(w)| \cdot |\psi'(0)| \\ &= \frac{1 - |\psi(f(w))|^2}{1 - |f(w)|^2} |f'(w)|(1 - |\varphi(0)|^2) \\ &= \frac{1}{(1 - |f(w)|^2)} |f'(w)|(1 - |w|^2), \end{aligned}$$

as wished. Equality holds at w if and only if $\varphi \circ f \circ \psi = \rho$ is a rotation, in which case $f = \varphi \circ \rho \circ \psi$ is an automorphism. We have so proved (ii).

Part (i) is proved using the same auxiliary automorphisms: $|(\varphi \circ f \circ \psi)(\zeta)| \leq |\zeta|$ by the first part of Schwarz' lemma, and letting $z = \psi(\zeta)$ (hence, $\zeta = \psi(z)$) we obtain $|(\varphi \circ f)(z)| \leq |\psi(z)|$, which is (3.11). Again, the equality case holds if and only if f is an automorphism. □

3.2.2. *The hyperbolic metric in the unit disc.* When treating Riemannian metrics we will here use old fashioned language and notation, since this suffices to our aims. You are however encouraged to translate everything in a more contemporary, rigorous, and far reaching language. We consider here the Riemannian metric ds^2 on $\mathbb{D}$ which is given by:

$$ds^2 = \frac{4|dz|^2}{(1 - |z|^2)^2}. \tag{3.13}$$

This metric was first introduced (in n real variables) by Riemann is his Habilitation lecture in 1854, and some years later it was studied in depth by Eugenio Beltrami. The geometry it induces on the unit disc is the *hyperbolic geometry*, and the model provided by the disc is generally named after Poincaré.

The metric (3.13) can be fruitfully compared with the spherical one we have seen earlier. The difference is that the factor $1 + |z|^2$ in the denominator is here replaced by $1 - |z|^2$. This means, as we shall see in a more quantitative way, that two points close to the boundary might look very close on the disc, but be very far away in the hyperbolic metric.

THEOREM 37 (iperbolic isometries). *The orientation preserving isometries of the metric* (3.13) *are the disc automorphisms.*

PROOF. The fact that automorphisms preserve the hyperbolic metric is a direct consequence of (26). If $w = \varphi(z)$, then

$$\frac{4|dw|^2}{(1 - |w|^2)^2} = \frac{4|\varphi'(z)dz|^2}{(1 - |\varphi(z)|^2)^2} = \frac{4|dz|^2}{(1 - |z|^2)^2}.$$

In order to show that there are no other isometries there are a number of ways, with long proofs, or requiring rather deep results. I sketch here an argument which is not too long and for which we already have developed most of the tools.

Observe that the metric (3.13) is *angle-conformal* to the Euclidean metric $|dz|^2$, i.e. that amplitude of angles between curves in $\mathbb{D}$ are the same in the Euclidean as in the hyperbolic geometry. A orientation preserving isometry η of $\mathbb{D}$, then, is an angle-conformal, orientation preserving map also with respect to the Euclidean geometry. We have proved that such maps are holomorphic. Hence, η is a bi-jective, holomorphic map of the disc onto itself, thus it is an automorphism. □

Alternatively, we might verify that the Möbius group acts transitively on the unit circle bundle in the tangent space of the hyperbolic disc, and then use a general result from geometry ensuring that we can not ask more from the group of the direct isometries of a manifold[2]. Or we might use geodesics and angles in order to find a Möbius maps which acts on points like a given direct isometry η. Perhaps there are still other lines of argument.

More important, the Schwarz-Pick lemma 36 (ii) can be reformulated in the following, suggestive way.

Proposition 10. *A holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is a contraction for the hyperbolic metric. That is, if d_h is the distance associated with the hyperbolic metric, then*

$$d_h(f(z), f(w)) \leq d_h(z, w). \tag{3.14}$$

At this moment, it is not obvious what are the equality cases to be considered in this context. We will return to this below. Surely, this proposition encourages those interested in holomorphic functions to learn more about the hyperbolic metric.

The *hyperbolic length* of a piecewise smooth curve $c : [a, b] \to \mathbb{D}$ is

$$\Lambda(c) := \int_c \frac{2|dz|}{(1-|z|^2)} = \int_a^b \frac{2|\dot{c}(t)|dt}{1-|c(t)|^2}. \tag{3.15}$$

The *hyperbolic distance* between $z, w \in \mathbb{D}$ is

$$d_h(z, w) = \inf_{c:c(a)=z, c(b)=w} \Lambda(c). \tag{3.16}$$

Theorem 38. (i) *The hyperbolic distance between z and w is*

$$d_h(z, w) = \log \frac{1 + \left|\frac{w-z}{1-\bar{w}z}\right|}{1 - \left|\frac{w-z}{1-\bar{w}z}\right|}. \tag{3.17}$$

(ii) *Geodesics are doubly infinite. They are circels or straight lines meeting $\partial\mathbb{D}$ at right angles.*

(iii) *Metric circles in the hyperbolic metric are Euclidean circles which do not intersect $\partial\mathbb{D}$. Viceversa, each such circle is a metric circle for the hyperbolic metric.*

Proof. We first compute the distance between 0 and $0 < r < 1$. Let $c : [0, 1] \to \mathbb{D}$ with $c(0) = 0$ and $c(1) = r$. Then, with $c(t) = z(t) = x(t) + iy(t)$,

$$\begin{aligned} \Lambda(c) &= \int_c \frac{2|dz|}{(1-|z|^2)} = \int_0^1 \frac{2|\dot{z}(t)|dt}{(1-|z(t)|^2)} \\ &\geq \int_0^1 \frac{2|\dot{x}(t)|dt}{(1-x(t)^2)} \end{aligned}$$

[2]It is a basic fact of physics that, once a geometric space is *homogeneous* and *isotropic*, no more symmetry can be asked of it (in terms of conservation of length). In geometric terms, it must have *constant sectional curvature*, and verify some global, topological constraints.

$$\begin{aligned} &\geq 2\int_0^r \frac{dx}{(1-x^2)} = \int_0^r \left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx \\ &= \log\frac{1+r}{1-r}, \end{aligned}$$

and the last term is achieved if $c(t) = rt$. We have than that

(i) $d_h(0,r) = \log\frac{1+r}{1-r}$;

(ii) the geodesic between 0 and r is the segment $[0,r]$ on the real line.

The statements on the formula for the distance and the shape of the geodesics in the general case follow by moving points around by means of automorphisms, and recalling that fractional linear maps send circles and straight lines to circles and straight lines.

Similarly, the circles centered at 0 in the hyperbolic metric are Euclidean circles, and moving them by automorphisms one proves that hyperbolic circles are Euclidean circles in general. The converse statement is intuitive "by continuity", but I do not know a very quick proof. See (iii) in exercise 27 below. □

EXERCISE 27. (i) *Complete the proof of (i) and (ii) in theorem 38.*

(ii) *Let* $r_n = 1 - 1/2^n$. *Show that* $d_h(r_n, r_{r+1}) \to \delta > 0$ *as* $n \to \infty$. *Find the precise value of* δ.

(iii) *Let* $0 < r < 1$ *and let* ρ. *Find* $-1 < a < r < 1$ *such that* $d_h(a,r) = \rho = d_h(r,b)$. *Use this to show the converse property in item (iii) of theorem 38.*

The function $s = h(t) = \log\frac{1+t}{1-t}$ $(0 \leq t < 1)$ is strictly increasing, convex, and $h(0) = 0$, hence $t = k(s) = h^{-1}(s)$ is (a) strictly increasing, (b) concave, (c) $k(0) = 0$.

EXERCISE 28. *Let* $k : [0,\infty) \to [0,\infty)$ *be a map satisfying (a-b-c), and let* d *be a distance on a set* X. *Then,* $\delta(x,y) = k(d_h(x,y))$ *defines a new, topologically equivalent distance on* X.

By the exercise, the *pseudo-hyperbolic distance* δ on $\mathbb{D}$,

$$\delta(z,w) := \left|\frac{w-z}{1-\bar{w}z}\right| \tag{3.18}$$

is in fact a distance on $\mathbb{D}$. It is often used in holomorphic function theory. Observe that $\mathbb{D}$ has diameter 1 with respect to the distance δ. It is a distance which is invariant under automorphisms, and this makes it very different from the Euclidean distance on $\mathbb{D}$.

EXERCISE 29. *We know that the completion of* $\mathbb{D}$ *with respect to the Euclidean metric is* $\mathbb{D}\cup\mathbb{T}$, *where* $\mathbb{T}$ *(the 1-**torus**) is the unit circle endowed with the Euclidean metric. Let* X *be the completion of* $\mathbb{D}$ *with respect to the pseudo-hperbolic metric, and* $X_0 = X \setminus \mathbb{D}$. *Show that* X_0 *is totally disconnected, and it is in fact isometric to* $\mathbb{T}$ *with the discrete metric:* $\delta(\zeta,\xi) = 1$ *if* $\zeta \neq \xi$.

EXERCISE 30. *Use the explicit formula for the hyperbolic distance to show that, if* f *is a holomorphis map from the unit disc into itself, and there are* $z \neq w$ *such that* $d_h(f(z), f(w)) = d_h(z,w)$, *then* f *is an automorphism.*

Holomorphic maps from the disc into itself are contractions in the weak sense: they satisfy a Lipschitz condition with constant $\lambda = 1$. Their behavior is typically very far from the contractions with $\lambda < 1$ which are the object of Banach's fixed point theorem. The simple example of $f(z) = z^2$ is already instructive. If $z \in \mathbb{D}$, then successive iterates of f move it towards 0, which is the behavior we expect from a *contractive* map: $f^{\circ n}(z) = (f \circ \cdots \circ f)(z) = z^{2^n} \to 0$ as $n \to \infty$. On the boundary of the disc, however, the map act as $f(e^{it}) = e^{2it}$, which is on the contrary *expansive*: f doubles the length of arcs on $\mathbb{T}$.

EXERCISE 31. *(i) Find an automorphism φ of $\mathbb{D}$ which has no fixed points in $\mathbb{D}$. (ii) Show that all automorphisms of $\mathbb{D}$ have fixed point in cl$\mathbb{D}$.*

A way to look at these sort of questions is provided, in fact, by *iteration theory*, which studies how the iterates of a given holomorphic map exhibits different behavior in different portions of the plane. A good starting point to learn *holomorphic dynamics* is [**Carleson, Gamelin**].

3.2.3. *The upper half plane model of hyperbolic geometry.* Everything we have said concerning the unit disc can be translated to the upper half-plane $\mathbb{C}_+ := \{z = x+iy : y > 0\}$ by means of the *Cayley map* $\kappa : \mathbb{D} \to \mathbb{C}_+$, which is the bi-holomorphic map defined by:

$$w = \kappa(z) = i\frac{1+z}{1-z}, \;\; z = \kappa^{-1}(w) = \frac{w-i}{w+i}. \tag{3.19}$$

The automorphism group $M(\mathbb{C}_+)$ is conjugate to that of the disc by κ, $M(\mathbb{C}_+) = \kappa \circ M(\mathbb{D}) \circ \kappa^{-1}$. There is a much more direct representation of $M(\mathbb{C}_+)$.

PROPOSITION 11. *The maps in $M(\mathbb{C}_+)$ are those having the form*

$$\varphi(z) = \frac{az+b}{cz+d}, \;\; \textit{with } \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1. \tag{3.20}$$

The group $M(\mathbb{C}_+)$ is among the most studied non-compact Lie groups, and it has another official name,

$$M(\mathbb{C}_+) = Sl(2, \mathbb{R}).$$

EXERCISE 32. *Prove proposition 11. There are many ways to do that: some short, some others full of calculations.*

We now translate the objects we have seen in the disc in upper half plane coordinates. When the proofs are merely direct calculations, their are left to you as an exercise.

(i) The hyperbolic metric on $\mathbb{C}_+$ is

$$ds^2 = \frac{|dz|^2}{y^2}, \tag{3.21}$$

where $z = x + iy$, $y > 0$. The pseudo-hyperbolic distance between z and w is

$$\delta(z, w) = \left|\frac{w-z}{\bar{w}-z}\right|, \tag{3.22}$$

and the hyperbolic distance becomes

$$d_h(z, w) = \log\frac{1 + \left|\frac{w-z}{\bar{w}-z}\right|}{1 - \left|\frac{w-z}{\bar{w}-z}\right|}. \tag{3.23}$$

(ii) The hyperbolic geodesics in $\mathbb{C}_+$ are Euclidean circles and straight lines which are normal to $\mathbb{R}$. The metric, hyperbolic circles in $\mathbb{C}_+$ are those Euclidean circles which do not intersect $\mathbb{R}$.

(iii) The linear group of the real line, having as elements $x \mapsto ax + b$ ($a, b \in \mathbb{R}$, $a > 0$), after replacing real x by $z \in \mathbb{C}_+$, can be thought of as a subgroup of $Sl(2, \mathbb{R})$. It has two interesting subgroups, which generate it: *translations* $z \mapsto z + b$, and *dilations* $z \mapsto az$ ($a > 0$). The whole group $Sl(2, \mathbb{R})$ is generated by translations, dilations, and the *inversion* $z \mapsto -1/z$.

(iv) We use dilations to obtain a geometric interpretation of *cones* $\Gamma_\alpha = \{re^{it} : |t - \pi/2| \leq \alpha, r > 0\}$, where $0 \leq \alpha < \pi/2$. Γ_0 is just a geodesic.

PROPOSITION 12. *There is a continuous, strictly increasing function R from $[0, \pi/2)$ onto $[0, \infty)$ such that*

$$\Gamma_\alpha = \{z : d_h(z, \Gamma_0) \leq R(\alpha)\}. \tag{3.24}$$

Equivalently,

$$\partial\Gamma_\alpha \setminus \{0\} = \{z : d_h(z, \Gamma_0) = R(\alpha)\}.$$

PROOF SKETCH. Let D_0 be a Euclidean disc with center at $i \in \Gamma_0$, which is tangent to $\partial\Gamma_\alpha$ (in two points: P_- on the left, and P_+ on the right). D_0 is also a hyperbolic disc of radius R, having hyperbolic center at a point λi (with $\lambda > 0$). We have that $d_h(i\lambda, P_\pm) = R$: the hyperbolic geodesic $\langle P_, - P_+\rangle$ has length $2R$, and it intersects at right angles Γ_0 at $i\lambda$. It easily follows from this that $d_h(P_\pm, \Gamma_0) = d_h(P_\pm, \lambda i) = R$. We use here the fact that geodesics issued from the center of a metric disc hit the boundary at right angles: a general fact known as *Gauss lemma*, that in the specific case can be checked for the hyperbolic discs centered at 0 in the disc, then extended via automorphisms, which are conformal.

Dilation fix γ_0 and Γ_α, they move D_0 to discs whose centers μi ranges all over Γ_0, and they move the points $P_\pm$ to points $Q_\pm$ ranging all over the two rays which bound Γ_α. Since dilations are hyperbolic isometries, $d_h(Q_\pm, \Gamma_\alpha) = d_h(P_\pm, \Gamma_\alpha) = R$. □

EXERCISE 33. *Find the relation $\alpha \mapsto R(\alpha)$.*

Moving the cone Γ_α of point in $\mathbb{C}_+$ by means of automorphisms, or moving it back to $\mathbb{D}$ using the Cayley map, we obtain an intrinsic, metric interpretation for lines circles in $\mathbb{D}$ (or $\mathbb{C}_+$), intersecting $\partial\mathbb{D}$ at an angle $\pi/2 - \alpha$. They play the role *parallels* play on the sphere (they keep constant from a geodesic), and we might give them the same name.

THEOREM 39. *Let γ be a circle meeting $\partial\mathbb{D}$ at an angle $\pi/2 - \alpha$ at two points ζ, ξ. Let $\langle \zeta, \xi \rangle$ be the geodesic with endpoints "at infinity" ζ and ξ. Then, the points of $\gamma \cap \mathbb{D}$ have a distance $R(\alpha)$ from $\langle \zeta, \xi \rangle$. The other points with the same property are on an analogous arc of a circle on the opposite side of $\mathbb{D}$ with respect to $\langle \zeta, \xi \rangle$.*

What we have seen in this item is that what our Euclidean eye perceives as a cone with a vertex at the boundary, the hyperbolic eye sees as a metric cylinder around a geodesic.

(v) The only straight lines and arcs of circles in $\mathbb{D}$ which are still without an interpretation in terms of hyperbolic geometry are the circles which are tangent to the boundary: the *horocycles*. After a Cayley map and possibly an automorphism, we might assume that the horocycle is the straight line $\omega_y = \{x + iy : x \in \mathbb{R}\}$: it touches $\partial\mathbb{C}_+$ at ∞, and it meets at right angles all geodesics issuing from ∞ (straight half-lines meeting $\mathbb{R}$ at right angles). This looks like a family of level sets for a function having gradient tangent to the geodesics issuing from infinity, which is of course a function of the imaginary part y of $z = x + iy$. We have just to find a geometric function (one expressing some metric property) which depends on y alone.

In hyperbolic geometry, the *Busemann functions* $\beta_{\gamma,P}$ are parametrized by an oriented geodesic γ tending to the "ideal point" ζ (think of $\zeta \in \partial\mathbb{C}_+$) and by a point P on γ:

$$\beta_{\gamma,P}(z) = \lim_{Q\to\zeta \text{ on } \gamma} [d_h(Q,z) - d_h(Q,P)]. \tag{3.25}$$

Let's compute the Busemann function in $\mathbb{C}_+$ for $\gamma = \Gamma_0$ (the one in (iv)), and $P = i$, setting $Q = it$:

$$\begin{aligned}
\lim_{t\to+\infty} [d_h(it,z) - d_h(it,i)] &= \lim_{t\to+\infty} \left[\log \frac{1 + \left|\frac{it-z}{it-\bar z}\right|}{1 - \left|\frac{it-z}{it-\bar z}\right|} - \log \frac{1 + \left|\frac{it-i}{it+i}\right|}{1 - \left|\frac{it-i}{it+i}\right|}\right] \\
&= \lim_{t\to+\infty} \log \frac{1 - \left|\frac{it-i}{it+i}\right|^2}{1 - \left|\frac{it-z}{it-\bar z}\right|^2} \\
&= \lim_{t\to+\infty} \log \frac{|it+i|^2 - |it-i|^2}{|it-\bar z|^2 - |it-z|^2} \\
&= \lim_{t\to+\infty} \log \frac{4t}{-2\mathrm{Re}(itz) + 2\mathrm{Re}(it\bar z)} \\
&= \log \frac{1}{y},
\end{aligned}$$

which is in fact constant on each horocycle ω_y, normalized to vanish on the horocycle containing i.

THEOREM 40. *Each horocycle ω touching ∂D at ζ is a level set of the Busemann functions based on geodesics ending at ζ.*

(vi) We mentioned above several times the points on the boundary of $\mathbb{D}$ (or of $\mathbb{C}_+$), without specifying how they fit in the hyperbolic picture, since they do not belong to the hyperbolic plane itself. Consider again the geodesic Γ_0, oriented in the direction of increasing imaginary part. The points $a + iy$ on the geodesic $\Gamma_0 + a$ (a real, fixed), for $y > C_\alpha$ belong to Γ_α, no matter how α is small. On the other hand, for all other oriented geodesics, from some point on they leave Γ_α, no matter how close α is to $\pi/2$.

Let's summarize. On the class $\mathcal{G}$ of the oriented geodesics parametrized by arclength define the relation

$\alpha \sim \beta$ if and only if there is $C > 0$ such that $d_h(\alpha(t), \beta(t)) \le C$ for $t \ge 0$.

Then, $\sim$ is an equivalence relation, and in our models we have that $\alpha \sim \beta$ if and only if they point to the same boundary point. Moreover, we have the following dichotomy: if $\alpha \sim \beta$, then $\lim_{t\to 0} d_h(\alpha(t), \beta(t)) = 0$; if $\alpha \not\sim \beta$, then $\lim_{t\to 0} d_h(\alpha(t), \beta(t)) = +\infty$.

Much of what we have seen about hyperbolic "non-Euclidean" geometry, and much more, had already been envisioned by Lobacevsky and Bolyay. They had shown that the curves keeping a constant distance to a geodesic are not geodesics (as, by the way, it happens on a sphere), and had spotted the elusive horocycles, playing the role of circles centered at infinity. They also figured the existence of a circle at infinity with peculiar metric properties. Since our senses are prisoners of a universe which is at least locally Euclidean, they had to rely on logics, faith in their geometry, and a powerful imagination (with no "images"). It was Beltrami who, studying in depth the disc model of Riemann, and producing others on his own, made all these non-Euclidean objects and phenomena visible and easy to grasp in a portion of the Euclidean plane.

3.2.4. *Some hyperbolic quantities and function spaces.* The *hyperbolic area form* in $\mathbb{D}$ is

$$dA_h(z) = \frac{4dxdy}{(1-|z|^2)^2}. \tag{3.26}$$

It is a special case of the general formula for the area form associated with a Riemannian metric. By the general theory, dA_h is invariant under automorphisms, and it is contracted by holomorphic maps $f : \mathbb{D} \to \mathbb{D}$. This fact, however, can be easily checked by a direct calculation which uses (2.50),

$$\begin{aligned} dA_h(f(z)) &= \frac{4\det(Jf(z))dxdy}{(1-|f(z)|^2)^2} \\ &= \frac{4\det(Jf(z))dxdy}{(1-|f(z)|^2)^2} = \frac{4|f'(z)|^2dxdy}{(1-|f(z)|^2)^2} \\ &\le \frac{4dxdy}{(1-|z|^2)^2}, \end{aligned}$$

and we have used one of the "magic inequalities" for maps from the disc into itself, with equality as some z if and only if equality holds identically, if and only if f is an automorphism.

Next, we might compute the *distortion* of holomorphic $f : \mathbb{D} \to \mathbb{D}$ with respect to the hyperbolic metric:

$$\delta_{h,h}f(z) = \frac{d_h(f(z+dz), f(z))}{d_h(z+dz, z)} = \frac{1-|z|^2}{1-|f(z)|^2}|f'(z)|, \tag{3.27}$$

and by Schwarz lemma in its invariant form

$$\delta_{h,h}f(z) \le 1,$$

with equality at some z if and only if f is an automorphism.

More often, in holomorphic theory one is interested in the distortion of $f : \mathbb{D} \to \mathbb{C}$, where $\mathbb{D}$ carries the hyperbolic metric, while $\mathbb{C}$ carries the Euclidean one.

$$\delta_{h,e}f(z) = \frac{|f(z+dz) - f(z)|}{d_h(z+dz, z)} = \frac{1-|z|^2}{2}|f'(z)|. \tag{3.28}$$

The *Bloch (semi) norm* of holomorphic $f : \mathbb{D} \to \mathbb{C}$ is:

$$\|f\|_{\mathcal{B},*} := \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| = 2\sup_{z\in\mathbb{D}} \delta_{h,e}f(z). \tag{3.29}$$

The *Bloch space* $\mathcal{B}(\mathbb{D})$ is populated by those f's for which $\|f\|_{\mathcal{B},*} < \infty$.

EXERCISE 34. *Prove that, if $h : \mathbb{D} \to \mathbb{D}$ is holomorphic, then $\|f \circ h\|_{\mathcal{B},*} \le \|f\|_{\mathcal{B},*}$, and that if h is an automorphism, then equality holds.*

We might combine differently our hyperbolic objects, to obtain new, different *conformally invariants* objects. For instance, the *Dirichlet (semi)-norm* of holomorphic $f : \mathbb{D} \to \mathbb{C}$ is

$$\|f\|^2_{\mathcal{D},*} = \frac{1}{\pi}\int_{\mathbb{D}}[\delta_{h,e}f(z)]^2 dA_h(z) = \frac{1}{\pi}\int_{\mathbb{D}}|f'(z)|^2 dxdy. \tag{3.30}$$

The Dirichlet space $\mathcal{D}(\mathbb{D})$ is defined in the obvious way.

EXERCISE 35. *Find statements analogous to those of exercise 34, but for the Dirichlet space, and prove them.*

The invariance of the Dirichlet semi-norm under automorphisms also descends from the interpretation of the Dirichlet seminorm in terms of areas, which we have seen in (2.51).

3.3. Conformal invariance of the hyperbolic metric. Let $\varphi : \Omega \to \mathbb{D}$, $w = \varphi(z)$, a conformal map. We can use it to move the hyperbolic metric from $\mathbb{D}$ to Ω, or, which is the same, to find a model for the same metric in Ω coordinates. The metric which is so defined on Ω has the length element

$$ds^2 = \frac{|dw|^2}{(1-|w|^2)^2} = \frac{|\varphi'(z)|^2|dz|^2}{(1-|\varphi(z)|^2)^2}, \tag{3.31}$$

which is conformal to the Euclidean metric $|dz|^2$ on Ω (angles are measured the same way, and "infinitesimal circles" for ds^2 are the same as they are for $|dz|^2$, although their "infinitesimal radii" are multiplied by the constant $\frac{|\varphi'(z)|}{1-|\varphi(z)|^2}$).

The hyperbolic metric, and the way it interacts with the Euclidean metric, can be used to study properties of functions defined on Ω. We will see a couple of examples later in the course.

4. Some easy pieces

4.1. Abel's theorem on summation of series. For $\zeta \in \partial\mathbb{D}$, the *Stolz angle* $\Gamma_M(\zeta) \subset \mathbb{D}$ with vertex at ζ and parameter M is

$$\Gamma_M(\zeta) = \{z \in \mathbb{D} : |\zeta - z| < M(1-|z|)\}. \tag{4.1}$$

EXERCISE 36. *Show that $\partial\Gamma_M(1)$ meets $\partial\mathbb{D}$ at an angle $\alpha \in (0, \pi/2)$, where α depends on M.*

THEOREM 41 (Abel's theorem on power series). *Suppose that $\sum_{n=0}^{\infty} a_n$ converges, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then, f converges in $D(0,1)$ and*

$$\lim_{\Gamma_M(1)\ni z\to 1} f(z) = \sum_{n=0}^{\infty} a_n. \tag{4.2}$$

PROOF. By theorem 10 on power series, if $\sum_{n=0}^{\infty} a_n$ converges, then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is at least 1.

We can subtract a constant a_0 to make $\sum_{n=0}^{\infty} a_n = 0$, and we will assume that such is the case. We sum by parts. If $s_n = a_0 + \cdots + a_n$, $s_{-1} = 0$,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}(s_n - s_{n-1})z^n \\ &= \sum_{n=0}^{\infty} s_n(z^n - z^{n+1}) = (1-z)\sum_{n=0}^{\infty} s_n z^n \\ &= (1-z)\sum_{n=0}^{N} s_n z^n + (1-z)\sum_{n=N+1}^{\infty} s_n z^n \\ &= I + II. \end{aligned}$$

If $N \geq N(\epsilon)$ is so large that $|s_n| \leq \epsilon$ for $n > N$, then

$$\begin{aligned} |II| &= |1-z| \cdot \left| \sum_{n=N+1}^{\infty} s_n z^n \right| \\ &\leq |1-z| \cdot \epsilon \sum_{n=N+1}^{\infty} |z|^n \\ &= \frac{|1-z|}{1-|z|}\epsilon \\ &\leq M\epsilon. \end{aligned}$$

For that choice of N,

$$\begin{aligned} |I| &\leq |1-z|(N+1)\max\{|s_n| : n \geq 0\} \\ &\to 0 \end{aligned}$$

at $z \to 1$. □

The restriction of the approach region to the Stolz angle can not be removed, unless we have further information on the function f. Since many examples will naturally emerge as a consequence of the theory which we will be developing further on (H^p theory), I will not provide tricky examples here.

The relation between cones with vertex on the boundary and hyperbolic geometry we saw in proposition 12 somehow "explains" Abel's theorem on summation of series. We will see more when we discuss *non-tangential convergence* for holomorphic functions in H^p spaces.

4.2. Normal families. A family $\mathcal{F}$ of holomorphic functions defined on an open set Ω is *normal* if all sequences $\{f_n\}$ in $\mathcal{F}$ have a subsequence which converges uniformly on compact subsets.

THEOREM 42 (Montel's theorem). *A family $\mathcal{F}$ of holomorphic functions defined on Ω is normal if and only if it is* **locally bounded***: for all compact subset K of Ω,*

$$\sup_{f\in\mathcal{F}} \max_{z\in K} |f(z)| \leq M_K < \infty,$$

with M_K possibly dependent on K.

PROOF. We first prove the "if" part. We start with a closed disc $\bar{D} = \mathrm{cl}(D(a,r)) \subset \Omega$. By Cauchy formula, if $|z-a|, |w-a| \leq r/2$ (for short, $z, w \in 1/2\bar{D}$), then

$$\begin{aligned}
|f(w) - f(z)| &= \left| \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{\zeta - w} d\zeta - \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \\
&\leq \frac{M_{\bar{D}}}{2\pi} \int_{|\zeta - a| = r} \frac{|z - w|}{|\zeta - w| \cdot |\zeta - z|} |d\zeta| \\
&\leq \frac{M_{\bar{D}} 4 \cdot 2\pi r}{2\pi r^2} |z - w| = c(r) |z - w|.
\end{aligned}$$

Cover K with finitely many discs $1/2D_1, \ldots, 1/2D_m$ having closure contained in Ω. For all f in $\mathcal{F}$ and $z, w \in K$, then

$$|f(w) - f(z)| \leq \max(c(r_1), \ldots, c(r_m)) |w - z|.$$

This shows that $\mathcal{F}$ is equicontinuous on K, and we can use Ascoli-Arzelà and find a sequence $\{f_n\}$ in $\mathcal{F}$ which converges uniformly on K.

But we want to have convergence on *all* compact subsets, then we have to "diagonalize" in some way. Let $K_1 \subset \cdots \subset K_n \cdots \subset \omega$ be an exhaustion of Ω by compact sets the union of whose interiors is the whole of Ω. Any compact subset of Ω is contained in some K_n. Construct a sequence as follows. Let $\{f_m^1\}$ be a sequence which converges uniformly on K_1, and set $g_1 = f_1^1$. Find then a subsequence $\{f_m^2\}$ of $\{f_m^1\}$ which converges on K_2: $f_m^2 = f_{j_m}^1$, and we just require $j_1 > 1$ and set $g_2 = f_1^2 = f_{j_1}^1$, with $j_1 > 1$. Iterate, and always pick $g_l = f_1^l$.

The sequence $\{g_l\}$ is a subsequence of all subsequences introduced in the construction, hence, g_l converges uniformly on all K_n's, hence on all compact sets.

Let's prove the viceversa by contradiction: there is K and there are points z_n in K and functions f_n in $\mathcal{F}$ such that $|f_n(z_n)| \to \infty$. By taking a subsequence, we can assume $z_n \to a \in K$. Suppose also that a subsequence of the f_n's (we rename it $\{f_n\}$ to save notation) converges uniformly to some f, which is uniformly continuous on K. Then,

$$|f(a)| \geq |f_n(z_n)| - |f(a) - f(z_n)| - |f(z_n) - f_n(z_n)|.$$

The first summand can be made arbitrarily large for n large, the second can be made arbitrarily small for n possibly larger, and the third can be made arbitrarily small further increasing n. Then, $|f(a)| = \infty$, a contradiction. □

We have an analogous statement for harmonic functions.

THEOREM 43 (Montel's theorem for harmonic functions). *A family $\mathcal{F}$ of harmonic functions defined on a domain Ω is normal (i.e. for any sequence and any compact K in Ω, there is a subsequence converging uniformly on K) if and only if it is* **locally bounded***: for all compact subset K of Ω,*

$$\sup_{u \in \mathcal{F}} \max_{z \in K} |u(z)| \leq M_K < \infty,$$

with M_K possibly dependent on K.

You might follow the strategy in the proof of theorem 42, using the Poisson kernel representation on discs instead of the Cauchy formula.

EXERCISE 37. *Prove theorem 43.*

4.3. A quick look at holomorphic ordinary differential equations's. In Advanced Calculus, or in a course on ODEs, you have certainly met *Picard's existence theorem*. The ingredients of the proof are (a) translating the differential equation into an integral equation by means of the fundamental theorem of calculus, (b) turning the integral equation into an iteration scheme, (c) applying to the latter Banach's fixed point theorem. The "specific" ingredient is the fundamental theorem of calculus, which we have at our disposal in the holomorphic setting as well. The proof of the theorem, and of its extensions, can be translated almost *verbatim* in the holomorphic world. Here is an example, with proof.

THEOREM 44 (Picard's theorem for complex ODEs). *Let $\Omega \subseteq \mathbb{C}^2$ be open, and $f = f(z,w) : \Omega \to \mathbb{C}$ be holomorphic in both variables and C^1, and let $(a,b) \in \Omega$. Then, there exist $cl(D(a,r) \times D(b,q)) \subset \Omega$, and $\varphi : clD(a,r) \to clD(b,q)$, such that:*

$$\begin{cases} \varphi'(z) = f(z,\varphi(z)) \\ \varphi(a) = b. \end{cases} \tag{4.3}$$

Moreover, the solution is unique: if $\varphi_1 : D(a,r_1) \to D(b,q_1)$ is another solution, then $\varphi_1(z) = \varphi(z)$ on $D(a, \min(r,r_1))$.

PROOF. Introduce the operator

$$T(\varphi)(z) = b + \int_a^z f(\zeta,\varphi(\zeta))d\zeta, \tag{4.4}$$

mapping holomorphic φ to holomorphic $T(\varphi)$, provided it is well defined. The function $z \mapsto f(z,\varphi(z))$ is in fact holomorphic by the Cauchy-Riemann equations, with

$$\frac{d}{dz} f(z,\varphi(z)) = \partial_z f(z,\varphi(z)) + \partial_\zeta f(z,\varphi(z))\varphi'(z).$$

By Volterra-Poincaré, its integral on a path contained in a disc only depends on the endpoints. We have to choose $r > 0$ small enough to have $T(\varphi)(z) \in \mathrm{cl}D(b,q)$ if $z \in \mathrm{cl}D(a,r)$ and φ has values in $\mathrm{cl}D(b,q)$. To this aim, we estimate:

$$\begin{aligned} |T(\varphi)(z) - b| &= \left| \int_a^z f(\zeta,\varphi(\zeta))d\zeta \right| \\ &\leq |z-a| \cdot \max\{|f(\zeta,\xi)| : (\zeta,\xi) \in \mathrm{cl}(D(a,r) \times D(b,q))\} \\ &\leq r \max\{|f(\zeta,\xi)| : (\zeta,\xi) \in \mathrm{cl}(D(a,r) \times D(b,q))\}, \end{aligned}$$

and the last expression can be made less that q if r is small enough.

We next observe that $T(\varphi) = \varphi$ (with φ holomorphic) if and only if φ is a solution of (4.3). If it is a solution, integrating both sides of the ODE between a and z (we can choose the path by the local Volterra-Poincaré) we obtain the fixed point relation $T(\varphi) = \varphi$. Viceversa, differentiating the fixed point relation we obtain the ODE.

We then define the recursive scheme $\varphi_0(z) = b$, $\varphi_{n+1}(z) = T(\varphi_n)(z)$. If we show that T is a contraction with respect to the uniform norm, then $\varphi_n \to \varphi$ as $n \to \infty$ in the uniform (sup) norm. By a corollary to the Morera theorem, φ turns out to be in fact holomorphic, so $\varphi = T(\varphi)$ is our solution.

The proof of the contraction property does not differ from the usual one. If $\varphi, \psi : \mathrm{cl}D(a,r) \to \mathrm{cl}D(b,q)$ are holomorphic,

$$|T(\varphi)(z) - T(\psi(z))| = \left| \int_a^z [f(\zeta,\varphi(\zeta)) - f(\zeta,\psi(\zeta))]d\zeta \right|$$

$$
\begin{aligned}
&= \left|\int_a^z \left(\int_{\psi(\zeta)}^{\varphi(\zeta)} \partial_w f(\zeta,\xi)\right) d\zeta\right| \\
&\leq \int_{[a,z]} |\varphi(\zeta)-\psi(\zeta)|\cdot|d\zeta| \\
&\quad \cdot \max\{|\partial_w f(\zeta,\xi)| : (\zeta,\xi)\in \mathrm{cl}D(a,r)\times \mathrm{cl}D(b,q)\} \\
&\leq \max\{|\varphi(\zeta)-\psi(\zeta)| : \zeta\in \mathrm{cl}D(a,r)\} \\
&\quad \cdot \max\{|\partial_w f(\zeta,\xi)| : (\zeta,\xi)\in \mathrm{cl}D(a,r)\times \mathrm{cl}D(b,q)\} r,
\end{aligned}
$$

which can be made smaller than 1 if r is chosen small enough.

Banach's fixed point theorem is applied to T acting on the space of the functions which are holomorphic in $D(a,r)$ and continuous on its closure, and having values in $\mathrm{cl}(D(b,q))$. □

At first sight, the only difference with the real case is the appeal to a corollary of Morera's theorem, which is needed to ensure that the limit function is holomorphic (in the real case, we gain a degree of differentiability from $T(\varphi)=\varphi$, in the complex case things are slightly more complicated). Actually, there a much more substantial difference, which is related to the irrotational/conservative matter. Consider for instance the innocent looking 1^{st} order, linear, homogeneous ODE

$$\varphi' - \frac{\varphi}{z} = 0,$$

with initial condition $\varphi(1)=0$. Its solution, $\varphi(z)=\log z$, can not be defined on $\mathbb{C}\setminus\{0\}$, the domain of the coefficient $\frac{1}{z}$. There is here a problem with extending the gluing procedure used to produce maximal solutions for ODEs in a real variable, and the problem is that we might have irrotational, but non-conservative functions.

A beautiful, classical, and useful chapter of holomorphic theory is that of the second order, linear, differential equations of a complex variable, which leads to the hypergeometric function. We will not go into that, but you can study the basic first facts of the theory in [**Ahlfors**], §8-4.

The following is an exercise in "proof transplanting".

Exercise 38. *State and prove existence and uniqueness theorems for Cauchy problems for holomorphic ODEs: (i) of any order; (ii) linear, and of any order. Find a topological condition implying that the solutions of (ii) extend to global solutions.*

The next is less vague.

Exercise 39. *Let $f:\Omega\to\mathbb{C}$, $a\in\Omega$, $f'(a)\neq 0$. It has a (local) holomorphic left inverse φ if $(f\circ\varphi)(z)=z$ (on some neighborhood). This suggests that by solving the Cauchy problem*

$$\begin{cases}\varphi'(z) = \frac{1}{f'(\varphi(z))}\\ \varphi(f(a)) = a\end{cases}$$

we have an alternative proof of the inverse mapping theorem 14. Fix the details.

4.4. The $\bar\partial$ equation and holomorphic functions arising from it. Here we consider the $\bar\partial$ *problem*

$$\bar\partial f(z) = m(z),\ z\in\mathbb{C},\ f(\infty)=0, \tag{4.5}$$

where the coefficient $m\in C^1_c(\mathbb{C})$ is smooth and has compact support. The problem can be solved for more general boundary conditions (here we just have $f(\infty)=$

$\lim_{z\to\infty} f(z) = 0$), and much less smooth data. In order to do that, you need distributions and/or Sobolev theory. However, here I just want to point out that Pompeiu's formula is related to the solution of an important differential equation, which we will meet again.

Theorem 45. *Let $m \in C^1_c(\mathbb{C})$. then, (4.5) has a unique solution $f \in C^1(\mathbb{C})$, which has the form*

$$f(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{m(w)}{z-w} dudv = k * m(z), \tag{4.6}$$

where $k(z) = \frac{1}{\pi z}$.

This statement is one of the way to express the fact that $\frac{1}{\pi z}$ is the *fundamental solution* of the $\bar\partial$ equation $\bar\partial f = m$. More general statements relax the assumptions on m, but the regularity of the solution f decreases, and the very definition of "solution" has to be extended. This is what is done in PDE theory.

Proof. The function f is well defined because m has compact support and k is locally integrable. Moreover, since m is C^1, we can differentiate under the integral, applying Pompeiu's formula in the last equality,

$$\begin{aligned}
\frac{\partial f}{\partial \bar z}(z) &= \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial m}{\partial \bar z}(z-w) f(w) dudv \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar\partial m(w)}{z-w} dudv \\
&= m(z).
\end{aligned}$$

The fact that $f(z) \to 0$ as $z \to \infty$ is a trivial estimate (or dominated convergence). For uniqueness, we use Liouville's theorem 21: if (4.5) had two solutions f_1, f_2, then $\bar\partial(f_1 - f_2) = 0$ would be holomorphic in $\mathbb{C}$, and bounded; hence $f_1 - f_2$ would be constant, but $f_1(\infty) = f_2(\infty)$ implies that the constant is zero. □

We summarize here what we have just seen. Let $m \in C^1_c(\mathbb{C})$. The function f defined by (4.6) is certainly $C^1(\mathbb{C})$, holomorphic on $\mathbb{C} \setminus \mathrm{supp}(m)$, and $f(\infty) = 0$. We can verify that it is holomorphic at ∞,

$$\begin{aligned}
f(1/z) &= \frac{1}{\pi}\int_{\mathbb{C}} \frac{m(w)z}{1-zw} dudv \\
&= \sum_{n=0}^{\infty} \frac{1}{\pi}\int_{\mathbb{C}} m(w)w^n dudv \cdot z^{n+1},
\end{aligned}$$

where we can exchange integral and geometric series if $|z| < \frac{1}{\inf\{|z-w| : w \in \mathrm{supp}(m)\}}$. The operator

$$[Tm](z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{m(w)}{z-w} dudv \tag{4.7}$$

is injective from $C^1_c(\mathbb{C})$ to the functions f which are holomorphic in a neighborhood of ∞, $f(0) = 0$, and f extends to a C^1 function on $\mathbb{C}_*$. In the opposite direction, any such function is Tm for some $m \in C_c(\mathbb{C})$. Moreover,

$$\bar\partial(Tm) = m,$$

if $m \in C^1_c(\mathbb{C})$.

The requirement that $m \in C^1$ seems unnatural. In part, this hypothesis is an artifact of the proof, which only uses Green's theorem. Sharper results can be obtained using weak solutions. We will mention some in the chapter on harmonic and sub-harmonic functions.

5. The Riemann mapping theorem and some of its applications

The Riemann mapping theorem states that any simply connected domain which is not the whole plane can be conformally mapped onto the unit disc. At this point in the course, "simply connected" is replaced by "Volterra-Poincaré", but the fact that the two notions are equivalent will easily follow, in fact, from the Riemann mapping theorem itself. Riemann's result is in many ways *the* central result in one-dimensional holomorphic theory. It also says that there are many holomorphic maps: the structure of holomorphic functions, in all its rigidity, allows a great variety of objects.

The history of the theorem's proofs is a fascinating subject[3], and the quest for a rigorous proof (Riemann's original argument was based on a principle in calculus of variations, which took many years afterwards to find rigorous statements and proofs) spans fifty years of important mathematics.

At the end of a long day, we will have the following equivalences.

THEOREM 46 (Riemann mapping theorem in many versions). *The following properties are equivalent for a domain Ω in $\mathbb{C}$.*

(a) $\Omega \neq \mathbb{C}$ *has the Volterra-Poincaré property.*
(b) $\Omega \neq \mathbb{C}$ *and for all* $\zeta \in \mathbb{C} \setminus \Omega$, $\log(z-\zeta)$ *is well defined in* Ω.
(c) *There is a conformal map (holomorphic bijection)* $f : \Omega \to \mathbb{D}$.
(d) $\Omega \neq \mathbb{C}$ *is* **simply connected** *in the sense of homotopy.*
(e) $\Omega \neq \mathbb{C}$ *and for all* $\zeta \in \mathbb{C} \setminus \Omega$, *the* **winding number** $n(\gamma, \zeta)$ *of any closed curve in* Ω *vanishes.*
(f) $\Omega \neq \mathbb{C}$ *is nonempty and connected.*

All properties, but (c), hold for $\Omega = \mathbb{C}$.

So far, we know that (a) $\Longrightarrow$ (b), $\log(z-\zeta) = \int_{z_0}^{z} \frac{dw}{w-\zeta} + \log(z_0 - \zeta)$, where the second summand on the right is any of the determinations of the complex number $z_0 - \zeta$.

5.1. The Riemann mapping theorem. My favorite proof of the Riemann mapping theorem is Osgood's one from the beginning of the XX Century, which is in the spirit of Riemann's argument (see the first few chapters of [**Courant**]). Here, however, we give the proof by Koebe. No proof I know is "elementary", and this is the most challenging result from the "theoretical minimum".

THEOREM 47 (Riemann mapping theorem for Volterra-Poincaré domains). *Let* Ω *be a domain in* $\mathbb{C}$, $\Omega \neq \mathbb{C}$, *which has the Volterra-Poincaré property, and fix* $a \in \Omega$. *Then, there is a unique conformal map* $f : \Omega \to \mathbb{D}$ *(bijective, holomorphic) such that* $f(a) = 0$ *and* $f'(a) > 0$.

This is (a) $\Longrightarrow$ (c) in theorem 46, but the proof only makes use of (b).

PROOF. Let $\mathcal{F}$ be the family of the holomorphic $g : \Omega \to \mathbb{C}$ such that:

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[3]See: On the history of the Riemann mapping theorem J Gray - Rend. Circ. Mat. Palermo (2) Suppl, 1994

(i) g is conformal onto its image;
(ii) $g(\Omega) \subseteq \text{cl}\mathbb{D}$ (hence, by the open mapping theorem, $g(\Omega) \subseteq \mathbb{D}$);
(iii) $g(a) = 0$ and $g'(a) > 0$.

We wish to prove that

(a) $\mathcal{F}$ is not empty;
(b) there is an f in $\mathcal{F}$ which maximizes the functional $g'(a)$;
(c) such f has the required properties.

Let's start with (a). Pick $b \in \mathbb{C} \setminus \Omega$, and consider the function $h(z) = \sqrt{z-b}$ (one of the two possible such functions), which exists because Ω is Volterra-Poincaré and $z - b \neq 0$ in Ω. It is easy to see that h is $1-1$ and that there are not $z_1, z_2 \in \Omega$ such that $h(z_1) = -h(z_2)$. Let's verify this. Since $|h(z)| = \sqrt{|z-b|}$, it suffices to verify the properties for $|z - b| = r > 0$ fixed, and we know $\{z : |z - b| = r\} \setminus \Omega \neq \emptyset$ (otherwise, the Volterra-Poincaré property would fail since $\int_{|z-b|=r} \frac{dz}{z-b} = 2\pi i \neq 0$ and $1/(z - b)$ is holomorphic in Ω). We can assume $r = 1$ and (otherwise, dilate) and suppose that one of the points in $\{z : |z - b| = r\} \setminus \Omega$ lies on the positive real axis (otherwise, rotate and translate). The intersection $\{z : |z - b| = 1\} \cap \Omega$ is the union of (at most countably many) open arcs I_j with endpoints $(e^i\alpha_j, e^{i\beta_j})$, with $H_j = (\alpha_j, \beta_j) \subseteq (0, 2\pi)$. Let arg be a the determination of the logarithm in Ω: for z in I_j, $\arg(z - b) \in H_j + 2\pi n_j$, with n_j integer. Then,

$$\arg \sqrt{z-b} = 1/2 \arg z - b \in 1/2 H_j + \pi n_j.$$

If $\arg \sqrt{z_1 - b} = \eta_1 + \pi n_{j_1} \in 1/2 H_{j_1} + \pi n_{j_1}$ and $\arg \sqrt{z_2 - b} = \eta_2 + \pi n_{j_2} \in 1/2 H_{j_2} + \pi n_{j_2}$, then

$$\arg \sqrt{z_1 - b} - \arg \sqrt{z_1 - b} = \eta_1 - \eta_2 + (n_1 - n_2)\pi,$$

which is a multiple of π if and only if $H_{j_1} = H_{j_2}$ and $\eta_1 = \eta_2$, i.e. if and only if $z_1 = z_2$.

By open mapping theorem, $h(\Omega) \supset D(h(a), \rho)$ for some positive ρ (we take it small enough so that $\rho < |h(a)|$), thus it does not intersect the disc $D(-h(a), \rho)$. We take advantage of this to squeeze Ω in the unit disc by post-composing h with a fractional linear map ψ sending $-h(a) \mapsto \infty$, $h(a) \mapsto 0$, and adjusting by a factor ensuring that the complement of $D(-h(a), \rho)$ is mapped into the closed unit disc,

$$\psi(w) = c\frac{h(a) - w}{w + h(a)},$$

with the requirements on c that

$$1 > |\psi(-h(a) + \rho e^{it})| = \frac{|c|}{\rho}|2h(a) - \rho e^{it}| \geq \frac{|c|}{\rho}|h(a)|,$$

and that $(\psi \circ h)'(a) > 0$. The map $g(z) = \psi(h(z))$ belongs to $\mathcal{F}$.

Let's see (b). The family $\mathcal{F}$ is normal, since the functions in it are uniformly bounded. Let $\{g_n\}$ be a sequence in $\mathcal{F}$ such that $g_n'(a) \to \sup\{g'(a) : g \in \mathcal{F}\}$, which we can assume to be uniformly convergent on compact sets by normality, $g_n \to f$. By the Cauchy formula for derivatives, $f'(a) = \sup\{g'(a) : g \in \mathcal{F}\} > 0$. We only have to prove that f conformally maps Ω into $\mathbb{D}$. Let $a_1 \neq a$ in Ω, and consider the functions $g(z) - g(a_1)$ on $\Omega \setminus \{a_1\}$, with $g \in \mathcal{F}$. Each of them does not vanish on $\Omega \setminus \{a_1\}$, and $f - f(a_1)$ is the uniform limit on compacta of a sequence of such functions. By Hurwitz theorem 23, either $f - f(a_1)$ identically vanishes on $\Omega \setminus \{a_1\}$, but it can not because f is not constant, or it has no zeros on $\Omega \setminus \{a_1\}$, hence $f(a) \neq f(a_1)$. Hence, f is injective. By uniform continuity, $f(\Omega) \subseteq \mathrm{cl}\mathbb{D}$, hence, $f(\Omega) \subseteq \mathbb{D}$.

We finally have to show (c). The idea is that, if f is not onto, we can proceed with a square root, and find $k \in \mathcal{F}$ with $k'(a) > f'(a)$, which is a contradiction. Here are the details. Suppose that $w_1 \notin f(\Omega)$, and consider the following sequence of maps:

$$f_1 : z \mapsto \frac{f(z) - w_1}{1 - \overline{w_1} f(z)} = w \mapsto \sqrt{w} = \zeta = F(z) \mapsto d\frac{\zeta - F(a)}{1 - \overline{F(a)}\zeta},$$

where d is a unimodular constant, chosen to have a positive derivative for the composition. Before doing the calculation, let's see the geometry. The first map is itself the post-composition of f with a disc automorphism ψ which maps w_1 to 0; the square root "squeezes" $\psi(f(\Omega))$ while remaining in $\mathbb{D}$; and the last map is another automorphism ensuring that in the overall composition a is mapped to 0. Following a in this journey,

$$a \mapsto -w_1 \mapsto \sqrt{-w_1} \mapsto 0.$$

Using the magic relations for the derivatives of the disc automorphisms,

$$\begin{aligned} |f_1'(a)| &= |f'(a)| \cdot (1 - |w_1|^2) \cdot \frac{1}{2\sqrt{|w_1|}} \cdot \frac{1}{1 - |w_1|} \\ &= |f'(a)| \frac{1 + |w_1|}{2\sqrt{|w_1|}} \\ &> |f'(a)|, \end{aligned}$$

and we have reached a contradiction.

Uniqueness is left as an exercise. □

At first sight the result in the last calculation seems wrong: the sequence of maps $f(z) \mapsto d\frac{\zeta - F(a)}{1 - \overline{F(a)}\zeta}$ moves from the unit disc to itself, and $f(a) = 0 \mapsto 0$, so Schwarz lemma would give a derivative which is less than one, not greater! The fact is that we can apply the square root only because these maps are defined on $f(\Omega)$, not on all of the unit disc. Basically, the square root squeezes the image of Ω, but at the same time it moves the image of a closer to the boundary, hence further away (in hyperbolic terms) from w_1. The calculation says that the overall effect is that the composite map is expansive at a.

EXERCISE 40. *Making use of the automorphism group of the disc, prove the uniqueness statement in the Riemann mapping theorem.*

5.1.1. *Riemann's mapping theorem and the index of a curve.* We record here some easy implications in theorem 46. We define the *index of a closed curve with respect to a point*, which will be discussed more in details in the section on homology.

DEFINITION 4. *Let a be a point in the plane, and γ a closed, piecewise regular curve which does not contain a. The* **index** *of γ with respect to a is:*

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - a}. \tag{5.1}$$

PROPOSITION 13 ((b) $\iff$ (e) and (c) $\implies$ (a) in theorem 46). *Let $\Omega \neq \mathbb{C}$ be a domain.*

(i) *The function* $\log(z-\zeta)$ *is well defined in* Ω *for all* $\zeta \in \mathbb{C}$ *if and only if* $n(\zeta,\gamma)=0$ *for all* $\zeta \in \mathbb{C}$ *and all closed curves in* Ω*..*
(ii) *If there exists a conformal map* $f : \Omega \to \mathbb{D}$*, then* Ω *satisfies the Volterra-Poincaré property.*

Proof. (i) follows from definitions. The Volterra-Poincaré property is enjoyed by $\mathbb{D}$, and it is invariant under diffeomorphisms. The map f^{-1} is a diffeomorphism. □

5.2. Homotopy of planar domains. In this course, our main (although not exclusive) interest in curves lies in the fact that irrotational 1-forms can be integrated on them, the forms of the form $f(z)dz$ (f holomorphic) being especially central to our discourse. We are, that is, interested in the functionals

$$L(\omega,\gamma)=\int_\gamma \omega;\ L(f,\gamma)=\int_\gamma f(z)dz, \text{ with } f \text{ holomorphic in } \Omega, \tag{5.2}$$

where γ is a curve in Ω. If we consider γ fixed, $f \mapsto L(\gamma,f)$ defines a linear functional in f. It is interesting to know if for two curves γ,δ belonging to a class of interest (we will specify at least two) $L(\gamma,\cdot)=L(\delta,\cdot)$ for all holomorphic functions. There are two main viewpoints. The first, *homotopy*, considers continuous deformations of the curves. The second, *homology*, investigates if two closed curves make up the (oriented) boundary of a region contained in Ω (in a rather more general framework, which is more flexible and easy to use). In this subsection, we consider the viewpoint of homotopy.

5.2.1. *Definition and first properties.* Let Ω be a domain in $\mathbb{C}$.

Definition 5. (i) *Let* $\alpha,\beta : [c,d] \to \Omega$ *be curves,* $\alpha(c)=\beta(c)=z_0$*,* $\alpha(d)=\beta(d)=z_1$*. They are* **homotopic relative to** Ω *if there is* $H : [0,1]\times[c,d] \to \Omega$ *continuous such that* $H(0,t)=\alpha(t)$*,* $H(1.t)=\beta(t)$*,* $H(s,c)=z_0$*, and* $H(s,d)=z_1$*.*
(ii) *Let* $\alpha,\beta : [c,d]\to\Omega$ *be two closed curves. They are* **homotopic relative to** Ω *if there is* $H : [0,1]\times[c,d]\to\Omega$ *continuous such that* $H(0,t)=\alpha(t)$*,* $H(1.t)=\beta(t)$*, and* $H(s,c)=H(s,s)$ *(i.e.* $t\mapsto H(s,t)$ *is closed).*

We often have curves α,β *which are not defined on the same "time" interval. In this case, we say that that they are* **homotopic after reparametrization** *if there are reparametrizations* α',β' *of* α *and* β *that are homotopic. We will often forget the "after reparametrization" clause.*

We write $\alpha \sim_{\Omega,ht} \beta$ *if the curves* α *and* β *are homotopic in the sense (i) or (ii), omitting the subscript* Ω *when the domain is fixed,* $\alpha\sim_{ht}\beta$ *.*

We have given the definition which is "right" in topology, which has a minor drawback. We most often deal with α,β regular, but the intermediate curves $t\mapsto H(s,t)$ are just continuous because we did not ask more regularity of H. This accounts for some technicalities in some proofs.

Definition 6. *A closed curve* γ *in a domain* Ω *is* **contractible to the point** $a\in\Omega$ *(or* **homotopic to a constant***) if* $\gamma\sim_{ht}\gamma_a$*, where* $\gamma_a : [c,d]\to\Omega$*,* $\gamma_a(t)=a$ *is constant.*

A domain Ω *is* **simply connected** *if all closed curves in* Ω *are contractible to a point.*

The definition of simply connected is invariant under homeomorphisms.

LEMMA 3. *The disc $\mathbb{D}$ and the plane $\mathbb{C}$ are simply connected.*

PROOF. Disc and plane are dealt with in the same way. Let $\gamma : [c,d] \to \mathbb{D}$ (or $\mathbb{C}$) be a curve, and set

$$H(s,t) = s\gamma(t).$$

Then, $H : [0,1] \times [c,d] \to \mathbb{D}$ (or $\mathbb{C}$) is an homotopy between γ and γ_0. □

The Riemann sphere is simply connected as well, but there is a difficulty. If $\gamma : [c,d] \to \mathbb{C}_*$ is surjective, the elementary construction above can not be immediately transplanted. One can use Lebesgue lemma for open coverings of compact sets in metric spaces (see this entry in Mathematics Stack Exchange). If γ is not surjective, things are easy.

EXERCISE 41. (i) *Prove that a closed curve in $\mathbb{C}_*$ which is not surjective. Show that γ is contractible to a point.*
(ii) *Prove that a piecewise, smooth curve in $\mathbb{C}$ does not fill the plane.*

COROLLARY 14 ((c) $\Longrightarrow$ (d) in theorem 46). *If there is a conformal map $f : \Omega \to \mathbb{D}$, then Ω is simply connected.*

PROOF. The map $f^{-1} : \mathbb{D} \to \Omega$ is a homeomorphism. □

EXERCISE 42. *(*) We list here some elementary properties of homotopy, that you have most likely met in your Topology class.*

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5.2.2. *The proof that simply connected implies Volterra-Poincaré.* We prove here that (d) $\Longrightarrow$ (a) in theorem 46. The remaining implications (e) $\Longleftrightarrow$ (f), involving the condition that $\mathbb{C} \setminus \Omega$ be connected, will be deal with in the subsection on homology.

PROPOSITION 14. *Let Ω be a domain in $\mathbb{C}$. If Ω is simply connected, then it satisfies the Volterra-Poincaré property.*

The proof would be easier if our homotopy maps H were piecewise C^1. It can be proved, with not little effort, that if two regular curves are homotopic, then there is a piecewise linear map H realizing the homotopy (we can *regularize* the homotopy map). One could also require the intermediate curves to be piecewise linear. All this, however, requires effort which is not here needed. Instead, we *discretize* the homotopy map, and that's what we need for the proof. Here I follow the exposition in [**Tao**].

We prove a more general result.

THEOREM 48. *Let Ω be a domain, and ω be an irrotational 1-form, and let $\alpha, \beta : [c,d] \to \Omega$. be two piecewise regular curves in Ω.*

(i) *If α and β are closed and $\alpha \sim_{ht} \beta$, then $\int_\alpha \omega = \int_\beta \omega$.*
(ii) *If α and β have the same endpoints and $\alpha \sim_{ht} \beta$, then $\int_\alpha \omega = \int_\beta \omega$.*

Proposition 14 follows because, if Ω is simply connected, then two paths with the same endpoints are always homotopic, hence $\int_\alpha \omega = \int_\beta \omega$, which is one of the three equivalent definitions of the Volterra-Poincaré property.

PROOF. Properties (i) and (ii) are in fact equivalent, and we can just prove (ii) [the proof is anyway similar].

Let $H : [0,1] \times [c,d] \to \Omega$ be a homotopy betweenn $\alpha, \beta : [c,d] \to \Omega$, where $\alpha(c) = \beta(c) = z_0$ and $\alpha(d) = \beta(d) = z_1$. The map H is uniformly continuous,

$$|H(s,t) - H(s',t')| \leq \epsilon \text{ if } |s-s'| \leq \delta(\epsilon) \text{ and } |t-t'| \leq \delta(\epsilon), \tag{5.3}$$

and $H([0,1] \times [c,d])$ is compact in Ω, so its distance r to Ω is positive. Let δ be so small that (5.3) holds with $\epsilon = r/4$, and let $s_0 = 0 < s_1 < \cdots < s_m = 1$ and $t_0 = c < t_1 < \cdots < t_n = d$, where $s_j - s_{j-1} \leq \delta$ and $t_j - t_{j-1} \leq \delta$ for $j = 1, \ldots, n$.

If a, b are points in $\mathbb{C}$, let $\langle a, b\rangle$ be the oriented segments from a to be (i.e. $\langle a, b\rangle(\tau) = (1-\tau)a + \tau b$, $0 \leq \tau \leq 1$). For each $j = 2, \ldots, m-1$ and $i = 1, \ldots, n$, consider the closed quadrilateral

$$\begin{aligned} \gamma_{j,i} &= \langle H(s_j, t_{i-1}, H(s_j, t_i)\rangle + \langle H(s_j, t_i, H(s_{j-1}, t_i)\rangle \\ &\quad + \langle H(s_{j-1}, t_i, H(s_j, t_{i-1})\rangle + \langle H(s_{j-1}, t_{i-1}, H(s_j, t_{i-1})\rangle, \end{aligned}$$

which lies in $D(H(s_{j-1}, t_{i-1}), r) \subset \Omega$ because its perimeter is no more than r. For $j = 1$, replace the "lower side" $\langle H(0, t_j, H(s_j, t_{i-1})\rangle$ by the reversal of the restriction of α to $[t_{i-1}, t_i]$, and for $j = m$ replace the "upper side" $H(s_j, t_{i-1}, H(s_j, t_i)$ by the restriction of β to $[t_{i-1}, t_i]$: again, $\gamma_{j,i}$ lies in $D(H(s_{j-1}, t_{i-1}), r)$.

Since $D(H(s_{j-1}, t_{i-1}), r)$ satisfies Volterra-Poincaré and ω is irrotational, $\int_{\gamma_{j,i}} \omega = 0$ for all i's and j's. Summing over i, j, canceling integrals on sides which appear with opposite orientations, and taking into account that the terms with $i = 1$ and $i = n$ vanish because the homotopy keeps fixed the endpoints of the intermediate curves, we have:

$$\begin{aligned} 0 &= \sum_{i,j} \int_{\gamma_{j,i}} \omega \\ &= \sum_{i=1}^{n} \int_{\beta|_{[t_{i-i}, t_i]}} \omega - \sum_{i=1}^{n} \int_{\alpha|_{[t_{i-i}, t_i]}} \omega \\ &= \int_\beta \omega - \int_\alpha \omega. \end{aligned}$$

□

6. Homology of planar domains

6.1. The index of a curve. Let c be an anti-clockwise parametrization of the unit circle. We have computed

$$\frac{1}{2\pi i} \int_c \frac{dz}{z} = 1,$$

which also works as a counterexample for the Cauchy theorem on $\mathbb{C} \setminus \{0\}$.

Let γ be a closed, regular curve in $\mathbb{C}$, and a a point which does not lie on γ The *index of* γ *with respect to* a is:

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}. \tag{6.1}$$

The number $n(\gamma, a)$ is also called the *winding number of* γ *around* a.

Before we proceed, let's have an intuition for this definition of $n(\gamma, a)$. Its value will become soon evident, and it relies on the fact that $\frac{1}{z-a}$ lives in the holomorphic world. By translation invariance, we can take $a = 0$. After doing calculations,

$$n(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \left(\frac{xdx + ydy}{x^2 + y^2} + i \frac{ydx - xdy}{x^2 + y^2} \right) = \frac{1}{2\pi} \int_\gamma \frac{ydx - xdy}{x^2 + y^2}, \tag{6.2}$$

because $\frac{xdx+ydy}{x^2+y^2}$ is a *radial* 1-*form*, which is conservative,

$$\frac{xdx + ydy}{x^2 + y^2} = d(\log \sqrt{x^2 + y^2}) = d \log |z|.$$

To begin with, then, $n(\gamma, a)$ is real. The second 1-form, $\frac{ydx-xdy}{x^2+y^2}$, is the canonical example of an "irrotational but not conservative 1-form" from Vector Calculus. Actually, it is more that that:

$$\frac{ydx - xdy}{x^2 + y^2} = d \arg(z), \tag{6.3}$$

where arg is the argument, that can only be defined on $\mathbb{C} \setminus L$, L being a half-line starting at the origin. In fact, for $x \neq 0$, $\arg z = \arctan y/x + c$, which satisfies (6.3), and, in a neighborhood of $x = 0$, $\arg z = \arg(e^{-\pi/2} z) + c$, which again satisfies (6.3).

The punchline is that $\omega = \frac{ydx-xdy}{x^2+y^2}$ measures the variation of the argument of a point moving on the curve. Since the curve is closed, it must be an integer. But we are rigorous, so we will prove it by computation, and below you will find exercises making more precise in what the "variation of the argument" consists.

THEOREM 49. *Let γ be a closed curve which does not pass through a. Then, $n(\gamma, a)$ is an integer.*

PROOF. We consider the intermediate positions

$$n(t) = n(\gamma, a)(t) = \frac{1}{2\pi i} \int_\alpha^t \frac{\dot{z}(s)}{z(s) - a} ds, \tag{6.4}$$

where $z = \gamma(s)$ and $\gamma : [\alpha, \beta] \to \mathbb{C}$. Then,

$$\frac{d}{dt} e^{-2\pi i n(t)} = e^{-2\pi i n(t)} \frac{-\dot{z}(t)}{z(t) - a},$$

from which we have that

$$\frac{d}{dt} \left(e^{-2\pi i n(t)} (z(t) - a) \right) = e^{-2\pi i n(t)} \left(-\dot{z}(t) + \dot{z}(t) \right) = 0.$$

The function $e^{-2\pi i n(t)}(z(t) - a)$ is the constant, so $z(\alpha) - a = e^{-2\pi i n(t)}(z(t) - a)$:

$$e^{2\pi i n(t)} = \frac{z(t) - a}{z(\alpha) - a}, \tag{6.5}$$

which becomes obvious once you draw a picture. The curve is closed, hence $e^{2\pi i n(\beta)} = 1$, which shows that $n(\beta) = n(\gamma, a)$ is an integer. □

The Cauchy theorems we have so far say something about the vanishing of the index of a curve..

PROPOSITION 15. *The index $n(\gamma, a)$ vanishes if:*

(i) *γ lies in a disc D, and a is outside d;*

(ii) *γ lies in a Volterra-Poincaré domain, and a is outside Ω.*
(iii) *Ω is a bounded domain having as boundary the (regular) curve γ, and a is outside $cl\Omega$.*

There is a novelty: $n(\gamma, a)$ depends with continuity on a, hence we can draw some more general conclusions. If γ is a closed curve in $\mathbb{C}$, its complement $\mathbb{C}_*$ is open, and it has at most countably many connected components, only one of which contains ∞ (the *unbounded component*). We say that those are the *components in which γ divides $\mathbb{C}_*$*.

Proposition 16. *Let γ be a closed, regular curve in $\mathbb{C}$. Then, $n(a, \gamma)$ is constant on each of the components in which it divides the extended plane, and $n(\gamma, a) = 0$ on the unbounded component.*

Proof. The image of a connected set under a continuous map in connected, and in this case it is a connected subset of the integers, which can only contain one point. The second assertion follows from (i) in proposition 15, because γ is contained in a disc and a can be taken outside it. □

Exercise 43. *Let γ be a closed, regular curve which does not contain the origin, and define $\Pi\gamma(t) = \frac{\gamma(t)}{|\gamma(t)|}$. Show that the variation of the argument is the same on γ and $\Pi\gamma$, and that, if $\Pi\gamma$ is regular, then $n(\Pi\gamma, 0) = n(\gamma, 0)$.*

6.2. Chains and cycles. Chains and cycles are a handy way to deal with the topology of surfaces, their decompositions, the way different patches can be glued together along curves, the boundary of regions, and so on. Our viewpoint is that of line integrals, so we start from there.

A *(regular) cycle* in the plane is a formal linear combination

$$\Gamma = \sum_{j=1}^{n} m_j \gamma_j, \tag{6.6}$$

where m_j is an integer and γ_j is a piecewise regular curve. To each 1-form ω with continuous coefficients we can associate the map

$$\Gamma \mapsto \int_\Gamma \omega := \sum_{j=1}^{n} m_j \int_{\gamma_j} \omega. \tag{6.7}$$

Two chains Γ_1 and Γ_2 will be identified if $\int_{\Gamma_1} \omega = \int_{\Gamma_2} \omega$ for all 1-forms ω (you might formalize this as an equivalence relation). You can easily verify the following identifications.

(i) We identify a curve γ with a reparametrization of it.
(ii) We identify $-\gamma$ with a reversal of γ.
(iii) A constant curve γ_a is identified with 0.
(iv) The sum $\alpha + \beta = \gamma$ if γ is the juxtaposition of α and β.
(v) From (ii) and (iii), we can cancel a curve which is run in opposite directions.

Allowing repetitions and performing reversals for the negative coefficients, we can suppose that

$$\Gamma = \alpha_1 + \cdots + \alpha_m, \tag{6.8}$$

with no constant curves.

If in (6.7) the curves are $\gamma_j : [c_j, d_j] \to \Omega$, the *boundary* of the chain is:

$$\partial\Gamma = \sum_{j=1}^{n} m_j(\gamma_j(d_j) - \gamma_j(c_j)), \tag{6.9}$$

the formal linear combination of the endpoints of the curves, with signs as in the formula. We clearly have

$$\partial(\Gamma_1 + \Gamma_2) = \partial\Gamma_1 + \partial\Gamma_2.$$

A chain Γ is a *cycle* if $\partial\Gamma = 0$.

(i) A closed curve γ is a cycle.
(ii) The linear combination of closed curves is a cycle.
(iii) Viceversa, each cycle can be identified with a linear combination of closed curves.

We can prove (iii) recursively. Let C_0 be the family of the curves appearing in (6.8) and pick one of them, call it β_1. If $\partial\beta_1 = 0$, then β_1 is a cycle, and stop. Otherwise $\partial\beta_1 = b_1 - a_1$, with $a_1 \neq b_1$. By hypothesis, there must be a curve β_2 in the family which starts at b_1, otherwise the point b_1 is not canceled: call it β_2, and observe that $\partial(\beta_2 + \beta_2) = b_2 - a_1$, where b_2 is the final endpoint of β_2. If $b_2 = a_1$, then $\beta_1 + \beta_2$ is a cycle, otherwise proceed. Since there are just finitely many curves, there is l such that

$$b_l - a_1 = \partial(\beta_1 + \cdots + \beta_l) = 0,$$

hence $\beta_1 + \cdots + \beta_l$ is a cycle.

The remaining curves add to a cycle by additivity, and they are less. Iterate, and exhaust the list.

6.2.1. *Cycles arising as boundaries of unions of squares.* The best way to put cycles of curves into context is simplicial homology, where we consider "cycles" of topological triangles, rather than curves, whose boundaries are signed combinations of curves, rather than of points. This would lead us too far, however, and we do not really need this here. If you had previous exposure to simplicial homology, you can see by yourself how it fits here.

Instead, since we need it for a different characterization of simply connected regions, we see how cycles naturally arise a (oriented) boundaries of regions which are unions of finitely many squares. This also gives us a feeling of how the notions of boundary and of cycle interact. To make the feeling solid maths, you need simplicial homology.

Consider a mesh of (closed) squares with sides parallel to the axis, e.g. those whose vertices have integer coordinates, and for each square Q parametrize its boundary ∂Q in such a way the square is on the left of the boundary curve. Consider now a finite family $\mathcal{F}$ of such squares.

PROPOSITION 17. *There is a unique, disjoint decomposition $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_M$ such that, if $K_j = \cup_{Q \in \mathcal{F}_j} Q_j$,*

(i) *each $\overset{\circ}{K}_i$ (the interior of K_i) is connected;*
(ii) *the union of the interiors of several K_i's is not connected (that is, the $\overset{\circ}{K}_i$'s are the connected components of the interior of the union of the K_i's).*

Moreover, let $\Gamma_i = \sum_{Q\in\mathcal{F}_i} \partial Q$. *Then,* Γ_i *is a cycle which can be identified with the boundary of* K_i *(in which* K_i*) is always on the left. Two boundaries* Γ_i *and* Γ_j *can only intersect at points of the initial grid.*

The proposition is "obvious", but some elements of the proof have to be given.

Proof. Say that squares Q, Q' in $\mathcal{F}$ are related if there is a chain $Q = Q_0, Q_1, \ldots, Q_m = Q'$ such that Q_i and Q_{i+1} share an edge. It is an equivalence relation, and the equivalence classes $\mathcal{F}_j$ are easily seen to satisfy (i) and (ii).

To prove the statement concerning the cycle, draw pictures with the possible configurations of squares of $\mathcal{F}$ meeting at a common vertex, and inductively follow the boundary. There are only finitely many of them, hence the pictures provide rigorous proofs. □

6.3. A characterization of simply connected domains.

Theorem 50 ((e) $\iff$ (f) in theorem 46). *Let* Ω *be a domain. Then,* $\mathbb{C}_* \setminus \Omega$ *is connected if and only if* $n(\Gamma, a) = 0$ *for all* $a \notin \Omega$ *and all cycles* Γ *in* Ω.

A little thought shows that we might as well ask $n(\gamma, a) = 0$ for all closed curves, since each cycle can be identified with a closed curve by adding back-and-forth arcs. For the proof, we basically follow [**Ahlfors**] p.139-140.

Proof. If $\mathbb{C}_* \setminus \Omega$ is not connected, then it can decomposed as $A \cup B$, with A, B compact in the extended plane, disjoint, and A bounded. Let $a \in A$ and let $\delta > 0$ be the distance between A and B. We can consider the enlargement $A_{\delta/2} = \{z \in \mathbb{C} : d_E(z, A) \leq \delta/2\}$ of A, which is again compact and disconnected from B. Consider a mesh of isometric, closed squares Q with sides parallel to the axis, one of which, say Q_a, is centered at a. Choose their side to be small enough that no square of the mesh which intersects $A_{\delta/4}$ has points in common with a square intersecting B. To each square we can associate its boundary ∂Q, the sum of four oriented edges having Q on their left. Let now $\mathcal{F}$ be the family of the squares intersecting $A_{\delta/4}$, and let K be their union.

Consider the cycle $\Gamma = \sum_{Q\in\mathcal{F}} \partial Q$, where edges appearing with opposite orientations are eliminated. Each edge e appearing in Γ lies in Ω. In fact, if it touches A, then it has points of $A_{\delta/4}$ on both sides, hence it belongs to both mesh squares having it in common, and it was eliminated.

The square Q_a belongs to K, which might be made up of components surrounding each other and themselves in strange ways: that's the "maze". In order to escape it, we break its walls by moving vertically. Consider the strip Σ having the same horizontal side as Q_a. Then, $K \setminus \Sigma = R_1 \cup \cdots \cup R_k$ is the disjoint union of vertical rectangles, one of which (say R_1) contains Q_a. We have:

$$\begin{aligned} \int_\Gamma \frac{dz}{z-a} &= \int_{\Gamma - \sum \partial R_j} \frac{dz}{z-a} + \sum_j \int_{\partial R_j} \frac{dz}{z-a} \\ &= \int_{\Gamma - \sum \partial R_j} \frac{dz}{z-a} + 2\pi i, \end{aligned}$$

because

$$\int_{\partial R_j} \frac{dz}{z-a} = \begin{cases} 2\pi i \text{ if } j = 1, \\ 0 \text{ if } j > 1. \end{cases}$$

The cycle $\Gamma - \sum \partial R_j$ is (after elimination of edges with opposite orientations) free of edges meeting the vertical line through a, because all these edges have been eliminated by the boundaries of the rectangles. Thus, a is in the unbounded component of $\mathbb{C} \setminus \Gamma$, hence,

$$\int_{\Gamma - \sum \partial R_j} \frac{dz}{z-a} = 0.$$

In the end, we have found a cycle Γ so that $n(\Gamma, a) \neq 0$, as wished. $\square$

CHAPTER 2

Harmonic and sub-harmonic functions

We have met harmonic functions in their role of real/imaginary parts of holomorphic functions. Here we mostly consider harmonic functions *per se*. Harmonic functions are important in theory and applications to physics, and this is a first motivation. Second, they, and their non-homogeneous analogs (e.g. the *sub-harmonic functions*), also turn out to be an important tool in the study of holomorphic functions. Harmonic function theory can be extended to $\mathbb{R}^n$ for all $n \geq 2$, and they can be studied using tools coming from real, rather than complex analysis. Here we take advantage of holomorphic theory to much expedite proofs and calculations. Third, although they are strictly related with holomorphic functions (when $n = 2$), they have their own set of objects and phenomena. For instance, the minimal growth of a harmonic function at an isolated singularity is lower than the one in the holomorphic case. Still, it is possible to write down a dictionary translating back and forth many concepts from holomorphic to harmonic function theory. By the way, this suggests a number of generalizations and extensions, which are the subject of PDE theory and other branches of mathematics.

In the section we will also begin the study of *boundary values* for both holomorphic and harmonic functions in the unit disc (of course, similar problems arise for all domains), and we will see a couple of very interesting phenomena. (i) Any continuous function on the boundary of the disc extends to one that is harmonic inside the disc: this fails in the holomorphic case! It is possible, however, to characterize the functions on the boundary which have a holomorphic extension. (ii) It can happen that a harmonic function u can be extended with continuity to the closed disc, but its harmonic conjugate v can not! This opens the quest for "conjugation invariant" classes of harmonic functions: those defined by quantitative conditions (growth, regularity,...) that hold for u if and only if they hold for its conjugate v, if and only if they hold for the holomorphic $f = u + iv$. Continuity up to the boundary is not one such condition, although the request that f is continuous up to the boundary is very natural from other viewpoints. Important chapters of *harmonic analysis* started from the study of this kind of problems.

1. Harmonic vs. holomorphic functions

A function $u : \Omega \to \mathbb{C}$ is *harmonic* in a domain Ω if $u \in C^2(\Omega)$ and the *Laplace equation* holds,

$$\Delta u(z) = 4\bar{\partial}\partial u(z) = 0. \tag{1.1}$$

It is the homogeneous version of the *Poisson equation*,

$$\Delta u(z) = \mu(z), \tag{1.2}$$

where $\mu \in C(\Omega)$.

Holomorphic and harmonic functions are related in a number of ways. Before considering the general case, we briefly review the case of the polynomials in order to have a glimpse of what to expect.

1.1. Harmonic polynomials. Any polynomial p in the variables x, y can be written as a polynomial in $z, \bar{z}$:

$$p(z,\bar{z}) = \sum_{l,m} a_{l,m} z^l \bar{z}^m, \tag{1.3}$$

the coefficients $a_{l,m}$ vanishing for all but for a finite number of couples of indices. A polynomial like that in (1.3) vanishes if and only if all coefficients vanish.

EXERCISE 44. *Prove this fact. There is a two lines argument using linear algebra.*

The Laplacian of $p(z,\bar{z})$ is:

$$\bar{\partial}\partial p(z,\bar{z}) = \sum_{l\geq 1 \textbf{ and } m\geq 1} lm a_{l,m} z^{l-1}\bar{z}^{m-1},$$

which vanishes if and only if $a_{l,m} = 0$ whenever $l \geq 1$ **and** $m \geq 1$: we do not want monomials containing both z and $\bar{z}$.

PROPOSITION 18. *A polynomial is harmonic if and only if it has the form:*

$$p(z,\bar{z}) = \sum_l a_l z^l + \sum_m b_m \bar{z}^m = A(z) + B(\bar{z}), \tag{1.4}$$

where A and B are holomorphic polynomials.

We list some basic facts, whose proof is easy.

(i) The polynomial in (1.4) is real valued if and only if $\overline{a_l} = b_l$,

$$p(z,\bar{z}) = a_0 + \sum_{l>0} a_l z^l + \sum_{m>0} a_{-m}\bar{z}^m = \sum_{l\geq 0}(a_l z^l + \overline{a_l}\bar{z}^l).$$

(ii) The polynomial

$$\tilde{p}(z,\bar{z}) = \sum_{l>0} \tilde{a}_l z^l + \sum_{m>0} \tilde{a}_{-m}\bar{z}^m = \sum_l (\tilde{a}_l z^l + \overline{\tilde{a}_l}\bar{z}^l) = \sum_l (-i a_l z^l + i\overline{a_l}\bar{z}^l)$$

is harmonic and real valued for the same reasons and $p + i\tilde{p}$ is holomorphic,

$$(p + i\tilde{p})(z,\bar{z}) = 2\sum_l a_l z^l.$$

The polynomial $\tilde{p}$ is the *harmonic polynomial conjugate to* p, and its coefficients linearly depend on those of p in the following fashion:

$$\tilde{a}_l = \frac{1}{i}\begin{cases} a_l \text{ if } l > 0 \\ -a_l \text{ if } l < 0 \\ 0 \text{ if } l = 0. \end{cases}, \text{ i.e. } \tilde{a}_l = \frac{1}{i}\mathrm{sign}(l). \tag{1.5}$$

We will meet again this expression when we consider, more generally, the *conjugate function operator*.

In fact, the only real valued polynomials $p^\sharp$ with the property that $p + ip^\sharp$ is holomorphic are those which differ from $\tilde{p}$ by a real constant.

(iv) The *projection* (at least in the algebraic sense, but it can be framed in L^2 theory as well) of the harmonic polynomial p onto the space of the holomorphic polynomial is given by

$$p(z,\bar{z}) \mapsto \Pi p(z) = \sum_l a_l z^l = \frac{p+i\tilde{p}}{2}.$$

EXERCISE 45. *Let P_n be the complex linear space of the homogeneous polynomials of degree n. Compute the dimension of P_n, of the subspace of those polynomials which are harmonic, and that of the polynomials which are harmonic. Do the same calculation for the polynomials of degree $\leq n$.*

1.2. Harmonic functions.

We see how things work in general.

THEOREM 51. (i) *Holomorphic functions are harmonic.*

(ii) *If $f = u+iv$ is holomorphic, then u and v are harmonic. The level curves of u and v meet at right angles.*

(iii) *Let u be harmonic in a simply connecetd domain Ω, and let $a \in \Omega$. Then, there is a unique $v : \Omega \to \mathbb{C}$ such that $u+iv$ is holomorphic and $v(a) = 0$.*

($*$) Proof to be written.

When u, v are real valued harmonic functions on a domain Ω, and $u+iv$ is holomorphic, we say that v is *harmonically conjugate* to u (or, simply, if a normalization was imposed on v to guarantee its uniqueness, that v is the *conjugate harmonic function* to u).

COROLLARY 15. *Let u be harmonic in a domain Ω.*

(i) *[Mean Value Property] If u is harmonic in a domain Ω, and $cl(D(z_0,r)) \subset \Omega$, then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})dt. \tag{1.6}$$

(ii) *[Weak maximum and minimum principle] If u is harmonic and real valued in a domain Ω, and $a \in \Omega$ is local maximum, or minumum, for u, then u is constant.*

(iii) *[Holomorphic invariance] If $u : \Omega \to \mathbb{C}$ is harmonic, and $g : A \to \mathbb{C}$ is holomorphic with values in Ω, then $u \circ g$ is harmonic in A. More generally, if $u \in C^2$, then*

$$\Delta(u \circ g) = |g'|^2 \cdot (\Delta u) \circ g. \tag{1.7}$$

($*$) Proof to be written.

EXERCISE 46. *With the same hypothesis of corollary 15 (ii) (with $z_0 = 0$), show that we have the "solid" mean value property:*

$$u(0) = \frac{1}{\pi r^2}\int_{cl(D(z_0,r))} u(x+iy)dxdy.$$

Another immediate consequence of theorem 51 is that harmonic functions can be locally expended in power series. We state this result for the unit disc.

THEOREM 52 (Power series representation of a function harmonic in $\mathbb{D}$). *Let u be harmonic in $\mathbb{D}$. Then, there is a sequence $\{a_n\}_{n=-\infty}^{+\infty}$ such that both $\{a_n\}_{n=-\infty}^{+\infty}$*

and $\{a_n\}_{n=-\infty}^{+\infty}$ *have radius of convergence* $R \geq 1$*, and*

$$u(re^{it}) = \sum_{n=-\infty}^{+\infty} a_n r^{|n|} e^{int}. \tag{1.8}$$

The series converges totally for $|z| \leq r < 1$*.*

Viceversa, if, then (1.8) *defines a harmonic function. The coefficients* a_n *are uniquely determined by* u*.*

1.3. Mean value property, harmonicity, and the maximum principle. The mean value property in corollary 15 (i) characterizes in fact harmonic functions, and it it worth spending paying more attention to it. Also, the deduction of the maximum principle from the mean value property is a simple reasoning in statistics, which deserves to be written down in details. Let Ω be a domain in $\mathbb{C}$. The class $MVP(\Omega)$ (we will prove it coincides with that of the real valued, harmonic functions) contains the continuous functions $u : \Omega \to \mathbb{R}$ such that, for all z_0 in Ω, there is $r_0 > 0$ such that (1.6) holds for all $0 < r < r_0$.

Lemma 4 (Weak maximum principle from mean value property). *If* $u \in MVP(\Omega)$ *and* z_0 *is a point where* u *achieves its maximum, then* u *is constant in* Ω*.*

Corollary 16. *If* $u \in MVP(\Omega)$ *and* $u \in C(cl\Omega)$*, then the maximum* M *and the minimum* m *of* u *are achieved on the boundary of* Ω *(and not in the interior).*

Proof. Let $K = \{z_0 \in \Omega : u(z_0) = M\}$ is closed in Ω. We show it is open. Let $z_0 \in K$, and let $r_0 > 0$ as in the definition of $MVP(\Omega)$, small enough that $u(z) \leq M$ if $|z - z_0| < r_0$. It there were $0 < r < r_0$ and t such that $u(z_0 + re^{it}) \leq M - 2\eta$ (some $\eta > 0$), by continuity $u(z_0 + re^{is}) \leq M - \eta$ if $|s - t| \leq \delta$ (some $\delta > 0$), so

$$\begin{aligned} M &= u(z_0) = \frac{1}{2\pi}\left(\int_{|s-t|\leq\delta} + \int_{|s-t|>\delta}\right) u(z_0 + re^{is})ds \\ &\leq \frac{(M-\eta)2\delta + M(2\pi - 2\delta)}{2\pi} < M, \end{aligned}$$

a contradiction. Hence, K is open in Ω, thus $K = \Omega$. □

Since harmonic functions belong to MVP, we have proved (ii) in corollary 15. We can now remove the $MVP(\Omega)$ notation.

2. The Dirichlet (boundary) problem for the Laplace equation in the unit disc

In this subsection we show that, given a continuous function on the boundary of the unit disc, there is a unique harmonic function in the disc which attains those values. But first, let's review some notation.

The *torus* is $\mathbb{T} = \{e^{it} : t \in \mathbb{R}\} = \partial\mathbb{D}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc in the complex plane. We identify $e^{it} \in \mathbb{T} \longleftrightarrow t \in (-\pi, \pi]$ and functions on $\mathbb{T}$ with 2π-periodic functions on $\mathbb{R}$. In particular, the continuous function $f(e^{it})$ is identified with a continuous, 2π-periodic function $f(t) = f(t + 2\pi)$ defined on $\mathbb{R}$. The *convolution* of two functions $f, g : \mathbb{T} \to \mathbb{C}$ is here defined as

$$(f * g)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t - s)g(s)ds. \tag{2.1}$$

Observe that $-2\pi < t-s \leq 2\pi$: this is not a problem, after we have extended f to a 2π-periodic function, $f(t+2\pi) = f(t)$, which can be done in a unique way. On $\mathbb{T}$ we consider normalized arclength measure $\frac{dt}{2\pi}$.

Let $f \in C(\mathbb{T})$. The *Dirichlet problem* for the equation $\Delta u = 0$ in $\mathbb{D}$ with boundary data f consists in finding $u \in C^2(\mathbb{D}) \cap C(\mathrm{cl}\mathbb{D})$ such that:

$$\begin{cases} \Delta u = 0 \text{ in } \mathbb{D}, \\ u(e^{it}) = f(e^{it}) \text{ on } \mathbb{T}. \end{cases} \tag{2.2}$$

We will in fact write down a formula for the solution.

THEOREM 53. *Let $f \in C(\mathbb{T})$. The Dirichlet problem (2.2) has a unique solution u, which is given by:*

$$u(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i(t-s)})\frac{1-r^2}{|1-re^{is}|^2}ds, \tag{2.3}$$

with $re^{it} \in \mathbb{D}$.

The function in (2.3) is the *Poisson extension* of f to $\mathbb{D}$.

It is interesting and useful to consider Dirichlet problems with data in different classes: L^p functions, or even Borel measures, or distributions. We will see some extensions in the chapter on H^p spaces.

2.1. The Poisson kernel. For $0 \leq r < 1$ and $t \in \mathbb{T}$, let

$$P_r(t) = \sum_{n=-\infty}^{+\infty} r^{|n|}e^{int} \tag{2.4}$$

be the *Poisson kernel*, which can also be thought of as a function of $z = re^{it}$, $P_r(e^{it}) = P(z)$.

Here are some basic properties of the Poisson kernel, which it shares with the *approximations of the identity* (sometimes called *Friedrich mollifiers*) you might have met is some course on PDEs.

LEMMA 5. *The Poisson kernel satisfies:*

(i) $P_r(t) > 0$;
(ii) $\int_{-\pi}^{+\pi} P_r(t)\frac{dt}{2\pi} = 1$;
(iii) *for* $\delta > 0$,
$$\lim_{r\to 1} \sup_{\delta \leq |t| \leq \pi} P_r(t) = 0.$$
(iv) $P_r(e^{it})$ *is a harmonic function of* $z = re^{it}$ *in* $\mathbb{D}$.

PROOF. Integrating the series term by term we have (ii). An alternative expression for $P_r(t)$ can be obtained by computing the series, also proving (ii). Let $z = re^{it}$:

$$\begin{aligned} P_r(t) &= \sum_{n=0}^{\infty} z^n + \sum_{n=1}^{\infty} \overline{z}^n \\ &= \frac{1}{1-z} + \frac{1}{1-\overline{z}} - 1 \\ &= \frac{1-\overline{z}+1-z-(1-\overline{z})(1-z)}{|1-z|^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1-|z|^2}{|1-z|^2} \\
&= \frac{1-r^2}{1-2r\cos(t)+r^2} > 0.
\end{aligned}$$

From the closed expression for P_r we have:

$$\lim_{r\to 1} \sup_{\delta\le|t|\le\pi} P_r(t) = \lim_{r\to 1} \frac{1-r^2}{1-2r\cos(\delta)+r^2} = 0.$$

About statement (iv), $\frac{1}{1-z}$ is harmonic, hence is harmonic:

$$2\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1}{1-z} + \frac{1}{1-\overline{z}} = P_r(e^{it}) + 1,$$

showing that $P_r(e^{it})$ is harmonic. □

Denote $\tau_s f(e^{it}) = f(e^{i(t-s)})$, translation by s on $\mathbb{R}$ (or, rather, rotation by t radians in $\mathbb{T}$).

LEMMA 6 (Continuity of translations in $C(\mathbb{T})$). *Let $f \in C(\mathbb{T})$. Then,*

$$\lim_{s\to 0} \|\tau_s f - f\|_{C(\mathbb{T})} = 0.$$

PROOF OF THE LEMMA. Since $\mathbb{T}$ is compact, f is uniformly continuous, hence $\|\tau_s f - f\|_{C(\mathbb{T})} = \max\{|f(e^{it}) - f(e^{is})|\} \le \epsilon$ provided that $|t-s| \le \delta(\epsilon)$. □

2.2. Proof of the Poisson extension theorem.

PROOF OF THEOREM 53. The function $re^{it} \mapsto P_r(e^{i(t-s)})$ is harmonic, and the hypothesis for differentiating under the integral in (2.3) hold, if $r < 1$, hence, $\Delta u = 0$ in $\mathbb{D}$. We have to show continuity of u at boundary points e^{it_0}. We have

$$|u(re^{it}) - f(e^{it_0})| \le |u(re^{it}) - u(re^{it_0})| + |u(re^{it_0}) - f(e^{it_0})| = I + II.$$

For the first summand, we use lemma 6 and properties of the Poisson kernel.

$$\begin{aligned}
I &\le \frac{1}{2\pi}|f(t-s) - f(t_0-s)|P_r(s)ds \\
&\le \|\tau_{t-t_0} f - f\|_{C(\mathbb{T})} \frac{1}{2\pi} P_r(s)ds = \|\tau_{t-t_0} f - f\|_{C(\mathbb{T})} \\
&\le \epsilon
\end{aligned}$$

if $|t-t_0| \le \delta(\epsilon)$. For the second term,

$$\begin{aligned}
II &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s) \frac{1-r^2}{|1-re^{is}|^2} ds - f(t) \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t-s) - f(t)) \frac{1-r^2}{|1-re^{is}|^2} ds \right| \\
&\le \frac{1}{2\pi} \left(\int_{|s|\le\delta} + \int_{|s|>\delta} \right) |f(t-s) - f(t)| \frac{1-r^2}{|1-re^{is}|^2} ds \\
&\le \sup_{|s|\le\delta} \|\tau_s f - f\|_{C(\mathbb{T})} + \frac{2\|f\|_{C(\mathbb{T})}}{2\pi} \int_{|s|>\delta} \frac{1-r^2}{|1-re^{is}|^2} ds.
\end{aligned}$$

For fixed $\delta < \delta(\epsilon)$ the first summand is less than ϵ. We can then choose $\eta > 0$ such that if $1 - r < \eta$, then the second summand is less than ϵ, too. We have proved that (2.3) solves the problem.

Suppose there is another solution v. Then $u-v$ is harmonic in $\mathbb{D}$, continuous on cl($\mathbb{D}$), and it has boundary values zero. If it does not vanish identically, $u-v$ has a maximum or a minimum inside. By the maximum principle, it is constant. Contradiction. □

We will see below that the *Poisson extension* can be fruitfully written in terms of the Fourier coefficients of f. In the chapter on H^p spaces we will also see variations on the proof of theorem 53 which cover the case of boundary values in L^p ($1\le p\le\infty$). The big problem is understanding in which way they are "boundary values".

We considered here the unit disc. Using conformal invariance of the harmonic functions (i.e. by the "right" change of variables) we can solve the Dirichlet problem in general discs, and in other domains.

EXERCISE 47. *Show that the solution* $u\in C(clD(a,R))$ *of the Dirichlet problem*

$$\begin{cases}\Delta u=0 \text{ in } D(a,R),\\ u(a+Re^{it})=\varphi(e^{it}),\end{cases}$$

with $\varphi\in C(\partial D(a,R))$ *is given by:*

$$u(a+re^{it})=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\varphi(a+Re^{is})\frac{R^2-r^2}{|R-re^{i(t-s)}|^2}ds. \tag{2.5}$$

Observe that $|R-re^{i(t-s)}|^2=R^2-2rR\cos(t-s)+r^2$.

EXERCISE 48. *Let* $\varphi:\mathbb{R}\to\mathbb{R}$ *be continuous, bounded, and let*

$$u(x+iy)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\varphi(t)\frac{y}{(x-t)^2+y^2}dt,$$

with $x+iy\in\mathbb{C}_+$*, the upper-half plane.*

Show that

(i) $\Delta u=0$ *in* $x+iy\in\mathbb{C}_+$*;*
(ii) $\lim_{x+iy\to t}u(x+iy)=\varphi(t)$*;*
(iii) u *is bounded.*

(∗) To show uniqueness one can use reflection and Liouville property for harmonic functions. The latter follows from the power series expansion, for instance.

2.3. The conjugate function operator on the disc. Let u be harmonic in $\mathbb{D}$ and continuous on cl$\mathbb{D}$, real valued. We see now how to recover v, the conjugate harmonic function to u, and $u+iv$.

THEOREM 54. *Let* φ *be continuous and real valued on* $\mathbb{T}$*, and let* u *be its harmonic (Poisson) extension to the disc. Then,*

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\varphi(e^{is})\frac{e^{is}+z}{e^{is}-z}ds \tag{2.6}$$

defines a function which is holomorphic in $\mathbb{D}$*, such that* $Ref=u$ *is the harmonic extension to* $\mathbb{D}$*.*

The conjugate harmonic function v *to* u*, normalized with* $v(0)=0$*, is*

$$v(z)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\varphi(e^{is})\frac{2r\sin(t-s)}{|e^{is}-re^{it}|^2}ds \tag{2.7}$$

Proof. The function f defined by (2.6) is holomorphic in $\mathbb{D}$: take the derivative with respect to z under the integral, or by use Morera theorem and take instead the integral over a triangle inside the same integral. The *Cauchy kernel on the disc,*

$$C_z(e^{is}) = C(e^{-is}z) = \frac{e^{is}+z}{e^{is}-z} \tag{2.8}$$

comes from the Poisson kernel $P_z(e^{is}) = P(e^{-is}z)$:

$$\begin{aligned} P(z) &= \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \bar{z}^n - 1 = 2\sum_{n=0}^{\infty} \mathrm{Re}(z^n) - 1 \\ &= \mathrm{Re}\left(2\sum_{n=0}^{\infty} z^n - 1\right) \\ &= \mathrm{Re}\left(\frac{2}{1-z} - 1\right) \\ &= \mathrm{Re}\left(\frac{1+z}{1-z} - 1\right), \end{aligned} \tag{2.9}$$

from which (2.8) follows. By the Poisson representation of u,

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \varphi(e^{is})\mathrm{Re}\left(\frac{e^{is}+z}{e^{is}-z}\right) ds = \mathrm{Re}(f(z)).$$

Next, we observe that $C_0(e^{is}) = 1$ is real, hence, $v(z) = \mathrm{Im} f(z)$ satisfies the normalization $v(0) = \mathrm{Im} f(0) = 0$.

Finally, with $z = re^{it}$ we compute

$$\begin{aligned} \mathrm{Im}(C(z)) &= \mathrm{Im}\left(2\sum_{n=0}^{\infty} z^n - 1\right) \\ &= 2\mathrm{Im}\left(\sum_{n=1}^{\infty} r^n \sin(nt)\right) \\ &= \mathrm{Im}\left(1/i\sum_{n=0}^{\infty}(z^n - \bar{z}^n)\right) \\ &= \mathrm{Im}\left(1/i\left(\frac{1}{1-z} - \frac{1}{1-\bar{z}}\right)\right) \\ &= \mathrm{Im}\left(1/i\frac{z-\bar{z}}{|1-z|^2}\right) \\ &= \frac{2r\sin(t)}{|1-re^{it}|^2}. \end{aligned}$$

The relation (2.7) follows. □

The function

$$Q_z(e^{is}) = Q_{re^{it}}(e^{is}) = \frac{2r\sin(t-s)}{|e^{is}-re^{it}|^2} \tag{2.10}$$

is the *conjugate Poisson kernel* on $\mathbb{D}$. Its boundary values are more interesting than those of P_z (which are, in fact, a Dirac delta):

$$Q(e^{i(t-s)}) = Q_{e^{it}}(e^{is}) = \frac{2\sin(t-s)}{|e^{is}-e^{it}|^2} = \frac{\sin(t-s)}{1-\cos(t-s)} = \cot\frac{t-s}{2}. \tag{2.11}$$

Equation 2.7 suggests that, if φ is continuous and $u = P[\varphi]$ is its Poisson extension, then $Q * \varphi(e^{it})$ gives the boundary values of v, the harmonic conjugate to u. The answer is intricate and fascinating. One obstacle is that Q is not in $L^1(\mathbb{T})$, hence, defining the integral implicit in the convolution is delicate: one has to take the *principal value* of the integral, which might not exist for a subset of t's in the torus; and this leads to the theory of *singular integrals*, which is a cornerstone of contemporary analysis. The next obstacle is that v might not be continuous up to the boundary. It is easy to produce examples with a one-line proof using the Riemann mapping theorem. Since we do not have it yet, I will give a more direct example below. Its main interest is that it points to the good and the bad which is hidden in the conjugate kernel Q_z, and these good and bad play a crucial role in harmonic analysis.

2.3.1. *Continuity up to the boundary is not preserved by conjugation.* We exhibit a function u harmonic in the unit disc and continuous on its closure, whose harmonic conjugate is not continuous up to the boundary. Here are the heuristics. If $Q(e^{is}) = \cot(s/2)$ belonged in L^1, then $Q * \varphi$ would be continuous for φ continuous, but Q *is not* in $L^1(\mathbb{T})$: this is the bad feature of the kernel. On the other hand, $\int_{|s|>\epsilon} Q(e^{is})ds = 0$ for all $\epsilon > 0$ because $Q(e^{is})$ is odd: this is the good feature of the kernel. We want to construct a φ which is continuous, but it neutralizes the good feature. Our φ will be:

(i) odd, so that the cancellations of Q are of no use;
(ii) continuous;
(iii) $\int_{-\pi}^{\pi} \varphi(s)\cot(s/2)ds = +\infty$.

Requirement (iii) holds if, for instance, $\varphi(s) = 1/\log 1/s$ for $0 < s < \pi/2$. We can, if we wish, to make φ a C^1 function on $(-\pi, \pi] \setminus \{0\}$.

With such φ, we have that

$$v(1) = -\frac{1}{2\pi}\int_{-\pi}^{+\pi} \varphi(e^{is})\cot(s/2)ds = -\infty.$$

This is not wholly rigorous, because (2.7) defines v *inside* $\mathbb{D}$, not on its boundary. We clean the argument by computing, for $0 < r < 1$,

$$v(r) = -\frac{1}{2\pi}\int_{-\pi}^{+\pi} \varphi(e^{is})\frac{2r\sin(s)}{|e^{is}-r|^2}ds,$$

observing that $h(r) = \frac{2r\sin(t-s)}{|e^{is}-r|^2}$ is increasing for $0 < s < \pi$ (because $h'(s) > 0$), and deducing by monotone convergence that

$$\lim_{r\to 1} v(r) = v(1) = -\infty,$$

where $v(1)$ stands for the integral above. This is of course not compatible with the possibility of extending v to the boundary while preserving continuity.

3. Harmonicity and the mean value property

3.1. The converse of the mean value property.

THEOREM 55 (Inverse mean value property). *Let Ω be a domain in $\mathbb{C}$. Then, $MVP(\Omega) = h(\Omega)$, the space of the harmonic functions on Ω.*

PROOF. Since we are dealing with local properties, it suffices to show the result for $v \in MVP(D(z_0, r))$, which is also continuous on $\mathrm{cl}(D(z_0, r))$. Let u be the function which solves the Dirichlet problem for the Laplace equation in $D(z_0, r)$, having the same boundary values of v. Then, $u - v$ satisfies the mean value property, it is continuous in the open disc, hence it attains maximum and minimum at the boundary, where it vanishes. It follows that $v = u$ is harmonic in the disc. □

A not at all obvious consequence of the theorem is that continuity + mean value property $\Longrightarrow$ smoothness. Another consequence is Schwarz reflection principle, which we will prove shortly.

EXERCISE 49. *Use the mean value property to show that a real valued, non-constant harmonic function does not have isolated zeros.*

THEOREM 56. *Let u be a real valued harmonic function on a domain Ω. If u vanishes on an open (non-empty!) subset of Ω, then it vanishes identically.*

PROOF. Let $Z = \{z \in \Omega : u(z) = 0\}$, and suppose it contains a disc $D(z_0, r)$. Since $u = \mathrm{Re} f$ with f holomorphic in $D(z_0, r)$, by the open mapping theorem f, hence u, are constant on $D(z_0, r)$. Let $z \in \Omega$, and let γ be a curve in Ω starting at z_0 and ending at z. Since the trace of γ is compact, it can be covered by finitely many discs centered at point of γ. Also, γ is connected, hence there is a subsequence $D_0, \dots, D_n$ of these discs such that $z_0 \in D_0$, $z \in D_n$, and $D_{j-1} \cap D_j \neq \emptyset$. Since the intersections are open, we can repeatedly use the open mapping theorem, showing that $u(z) = 0$. □

EXERCISE 50. *Prove that a sequence of discs with the properties required in the proof of theorem 56 exists.*

COROLLARY 17 (Strong maximum principle). *Let u be harmonic in a domain Ω. If u has point of relative maximum in Ω, then u is constant. By theorem 56, it is constant on Ω.*

PROOF. By the weak maximum principle, if u has a relative maximum at $z_0 \in \Omega$, then it is constant in a disc $D(z_0, r)$. We can then use theorem 56. □

3.2. The Schwarz reflection principle.

THEOREM 57 (Reflection of harmonic functions). *Let u be harmonic in $\Omega \subseteq \mathbb{C}_+$, the upper half plane, and suppose that an open interval $I \subseteq \mathbb{R}$ lies on $\partial\Omega$, and that $\lim_{z \to x} u(z) = 0$ for all x in I. Consider the larger region $\widehat{\Omega} = \Omega \cup I \cup \overline{\Omega}$, and define $\widehat{u} : \widehat{\Omega} \to \mathbb{R}$,*

$$\widehat{u}(z) = \begin{cases} u(z) \ \text{if } z \in \Omega, \\ 0 \ \text{if } z \in I, \\ -u(\bar{z}) \ \text{if } z \in \overline{\Omega}. \end{cases} \tag{3.1}$$

Then, $\widehat{u}$ is harmonic in $\widehat{\Omega}$.

PROOF. The function $\widehat{u}$ is harmonic in $\Omega \cap \widehat{\Omega}$, hence it satisfy the local mean value property there. It also satisfies the mean value property at the points of I, by construction. By the inverse of the mean value property, $\widehat{u}$ is harmonic. □

THEOREM 58 (Reflection of holomorphic functions). *Let $f = u + iv$ be holomorphic in $\Omega \subseteq \mathbb{C}_+$. Let Ω, I be as in the previous theorem,* $\lim_{z\to x} v(z) = 0$*, and define* $\widehat{f} : \widehat{\Omega} \to \mathbb{R}$*,*

$$\widehat{f}(z) = \begin{cases} f(z) \text{ if } z \in \Omega, \\ u(x) \text{ if } z \in I, \\ \overline{f(\bar{z})} \text{ if } z \in \overline{\Omega}. \end{cases} \tag{3.2}$$

Then, $\widehat{f}$ *is holomorphic in* $\widehat{\Omega}$*.*

PROOF. The function $\widehat{f}$ is continuous on $\widehat{\Omega}$ and holomorphic in Ω. It also is holomorphic in $\overline{\Omega}$; e.g. by checking that it can be developed as a power series there. We can now use Morera's theorem, decomposing triangles in $\widehat{\Omega}$ which cross the real axis as union of smaller triangles, then approximating them with triangles contained in Ω and $\overline{\Omega}$. □

Using conformal maps $z = g(\zeta)$ and $w = \psi(\xi)$, we have versions of Schwarz result where the reflecting curve for $w = f(z)$ is not necessarily the real line. In fact, we can use a curve l to reflect z, and a different curve m to reflect w.

EXERCISE 51. *Write a statement of the reflection theorem in which the real line is replaced by a couple of any other straight lines in the plane.*

Here is a canonical example. Suppose $\Omega \subseteq \mathbb{D}$, with Ω having an open arc $I \subseteq \mathbb{T}$. Define $\Omega' = 1/\bar{z} : z \in \Omega\}$, and $\widehat{\Omega} = \Omega \cup \Omega' \cup I$. If $0 \in \Omega$, then $\infty \in \Omega'$.

PROPOSITION 19 (Reflection in the disc). *Let f be holomorphic in $\Omega \subseteq \mathbb{D}$, with cl$\Omega$ having an open arc $I \subseteq \mathbb{T}$. Suppose that* $\lim_{z\to\zeta} f(z) = \xi$ *exists for all ζ in I, and that $|\xi| = 1$. Define* $\widehat{f} : \widehat{\Omega} \to \mathbb{R}$*,*

$$\widehat{f}(z) = \begin{cases} f(z) \text{ if } z \in \Omega, \\ \lim_{w\to z} f(w) \text{ if } z \in I, \\ \overline{f(1/\bar{z})} \text{ if } z \in \Omega'. \end{cases} \tag{3.3}$$

Then, $\widehat{u}$ *is holomorphic in* $\widehat{\Omega}$*.*

PROOF. We give two proofs. The first one is short, but it does not explain how we came up with the map $z \mapsto 1/\bar{z}$. Since $z = 1/\bar{z}$ on $\mathbb{T}$ and $z \mapsto 1/\bar{z}$ exchanges the interior and the exterior of $\mathbb{T}$ in $\mathbb{C}_*$, (3.3) defines a function which is (i) continuous on $\widehat{\Omega}$, (ii) holomorphic in $\Omega \cup \Omega'$. Using Morera's theorem we can show that $\widehat{f}$ is holomorphic on $\widehat{\Omega}$. Let T be a triangle in $\widehat{\Omega}$ whose intersection with $\mathbb{T}$ lies in I, and it is so split by I into two figures $T_1 \subset \mathrm{cl}\Omega, T_2 \subset \mathrm{cl}\Omega'$. We can approximate T_1 by $T_1^\epsilon = T_1 \cap \mathrm{cl}D(0, 1-\epsilon)$ and T_2 by $T_2^\epsilon = T_2 \setminus D(0, 1+\epsilon)$. By Cauchy theorem

$$\begin{aligned} 0 &= \int_{\partial T_1^\epsilon} \widehat{f}(z)dz + \int_{\partial T_2^\epsilon} \widehat{f}(z)dz \\ &\to \int_{\partial T} \widehat{f}(z)dz, \text{ as } \epsilon \to 0, \end{aligned}$$

hence the hypothesis of Morera's theorem are verified.

The second proof gives an indication on how to find the reflecting map. Consider the conformal map $\varphi : z \mapsto w = i\frac{1-z}{1+z}$ which maps $\mathbb{D}$ onto the upper half plane

$\mathbb{C}_+$, having inverse $z = \frac{i-w}{i+w}$. The change of variables for the reflection $w \mapsto \bar{w}$ is

$$z \mapsto w = i\frac{1-z}{1+z} \mapsto \bar{w} = i\frac{\bar{z}-1}{\bar{z}+1} \mapsto \frac{i-\bar{w}}{i+\bar{w}} = \frac{1}{\bar{z}}.$$

Since φ and φ^{-1} are continuous up to the boundary, proposition 19 follows from theorem 58. □

Proposition 19 has some interesting consequences. We proved that any continuous function φ on $\mathbb{T}$ can be extended to a harmonic function in $\mathbb{D}$ having φ as boundary values. If we replace "harmonic" by "holomorphic" this is not true anymore. In fact, characterizing in an effective way which continuous functions on $\mathbb{T}$ admit a holomorphic extension to cl$\mathbb{D}$ is a tricky enterprise, and it is even more difficult to extract geometric, or analytic, properties these functions must have. We will meet this order of problems over and over.

A geometric feature of holomorphic functions is that they can not collapse a substantial portion of the boundary to a point. Here is a seminal result in this direction.

THEOREM 59. *Let f be holomorphic in a domain $\Omega \subset \mathbb{D}$, and suppose there is an arc $I \subset \mathbb{T}$ on $\partial\Omega$ such that $\lim_{z\to\zeta} f(z) = 0$ for all ζ in I. Then, f vanishes in Ω.*

PROOF. We can reflect f across $\mathbb{T}$, and obtain a function $\widehat{f}$ which is defined in $\widehat{\Omega} \supset I$, and such that $f(\zeta) = 0$ for $\zeta \in I$. Since I has accumulation points in $\widehat{\Omega}$, f vanishes identically. □

Another rewarding consequence of proposition 19 concerns *inner functions* which are continuous up to the boundary. We will meet inner functions in H^p theory.

COROLLARY 18. *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, such that $\lim_{z\to\zeta} f(z) = \xi$ exists for all $\zeta \in \mathbb{T}$, and $\xi \in \mathbb{T}$. Then, $\widehat{f}$ is a rational function.*

We will see at some point that f is in fact a *finite Blaschke product.*

PROOF. The function $\widehat{f}$ is holomorphic on the Riemann sphere, hence it is rational. □

A far reaching application of the reflection principle is the *Schwarz-Christoffel* method of constructing conformal maps from the disc to the interior of a polygon, which we will discuss in the chapter on conformal maps.

4. Digression: finite Blaschke products

For a survey of old and recent resultts concerning finite Blaschke products, see Finite Blaschke products: a survey by Stephan Ramon Garcia, Javad Mashreghi, and William T. Ross (MR3753897).

Recall that if $|a| < 1$, then $z \mapsto \frac{a-z}{1-\bar{a}z}$ defines an involutive automorphism of the unit disc. We normalize it to take value one at the origin:

$$b_a(z) = \frac{|a|}{a}\frac{a-z}{1-\bar{a}z}, \tag{4.1}$$

which we call a *Blaschke factor*. We set $b_0(z) = z$.

Definition 7. *A* **finite Blaschke product** B *is a finite product of Blaschke factors and of a unitary constant,*

$$B(z) = e^{is} z^m \prod_{j=1^n} \frac{|a_j|}{a_j} \frac{a_j - z}{1 - \bar{a_j} z}, \tag{4.2}$$

with $s \in \mathbb{R}$, $a_1, \dots, a_n \in \mathbb{D}$, *and* $m \geq 1$ *integer.*

The following facts follow from the definition.

(i) $v = B(0)$, m (the order of zero of $B(z)$ at $z = 0$), and $a_1, \dots, a_m$ (the other zeroes of B, each counted according to its multiplicity) are uniquely determined by B, which has the a unique expression in the form (4.2).
(ii) B maps $\mathbb{D}$ to $\mathbb{D}$, $\mathbb{T}$ to $\mathbb{T}$, and $\mathbb{C}_* \setminus (\mathrm{cl}\mathbb{D})$ to $\mathbb{C}_* \setminus (\mathrm{cl}\mathbb{D})$.
(iii) B has a pole of order m at ∞, and poles at $1/\bar{a_1}, \dots, 1/\bar{a_n}$.

5. Positive harmonic functions and Harnack's inequality

Harnack's inequality expresses in a quantitative way the fact that positive harmonic functions in a domain $\Omega \subset \mathbb{C}$ can not oscillate too much, and that the maximal oscillation "scales" with the distance to $\partial\Omega$. The basic estimate is in the following lemma. The oscillation has a multiplicative nature, as we shall see, not an additive one.

Lemma 7 (Basic Harnack's inequality). *Let* h *be harmonic and positive on* $D(a, r)$, *and* $0 < \rho < r$. *Then,*

$$\frac{r-\rho}{r+\rho} h(a) \leq h(a + \rho e^{it}) \leq \frac{r+\rho}{r-\rho} h(a), \tag{5.1}$$

and, in particular,

$$\max\{h(a + \rho e^{it}) : t \in \mathbb{R}\} \leq \left(\frac{r+\rho}{r-\rho}\right)^2 \min\{h(a + \rho e^{it}) : t \in \mathbb{R}\}. \tag{5.2}$$

The constants in the inequalities are best possible.

Proof. We can assume that h is harmonic on the closure of the disc: if (5.2) holds for all $r' < r$, it holds for r by continuity. Also, we can let $a = 0$. We use the Poisson representation of h and positivity,

$$\begin{aligned} h(\rho e^{it}) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} h(re^{i(t-s)}) \frac{r^2 - \rho^2}{|r - \rho e^{is}|^2} ds \\ &\leq \frac{r^2 - \rho^2}{(r-\rho)^2} \frac{1}{2\pi} \int_{-\pi}^{+\pi} h(re^{i(t-s)}) \frac{r^2 - \rho^2}{|r - \rho e^{is}|^2} ds \\ &= \frac{r+\rho}{r-\rho} h(0), \end{aligned}$$

by the mean value property. We can similarly estimate from below,

$$h(\rho e^{it}) \geq \frac{r^2 - \rho^2}{(r+\rho)^2} h(0) = \frac{r-\rho}{r+\rho} h(0).$$

Using the estimate from above for the maximum, and the one from below for the minimum,

$$\max\{h(\rho e^{it}) : t \in \mathbb{R}\} \leq \frac{r+\rho}{r-\rho} h(0) \leq \left(\frac{r+\rho}{r-\rho}\right)^2 \min\{h(\rho e^{it}) : t \in \mathbb{R}\},$$

as promised.

The fact that the estimate (5.2) is sharp can be seen by considering the "model" positive harmonic function, which is the Poisson kernel itself, $h(\rho e^{it}) = \frac{r^2-\rho^2}{|r-\rho e^{it}|^2}$. We have:

$$\begin{aligned} \max\{h(\rho e^{it}) : t \in \mathbb{R}\} &= \frac{r+\rho}{r-\rho} = \left(\frac{r+\rho}{r-\rho}\right)^2 \frac{r-\rho}{r+\rho} \\ &= \left(\frac{r+\rho}{r-\rho}\right)^2 \min\{h(\rho e^{it}) : t \in \mathbb{R}\}. \end{aligned}$$

□

Consider now the case of the unit disc $\mathbb{D} = D(0,1)$, then the disc $D(0,\rho) \subset \mathbb{D}$, and recall that

$$2\log\frac{1+r}{1-r} = \operatorname{diam}_h(D(0,\rho))$$

is the diameter of $D(0,\rho)$ in the hyperbolic metric. The inequality (5.2) can be rephrased in the following form: if h is positive, harmonic in $\mathbb{D}$, and $z, w \in \operatorname{cl}D(0,\rho)$, then

$$\frac{h(z)}{h(w)} \le e^{\operatorname{diam}_h(D(0,\rho))}. \tag{5.3}$$

We now use conformal invariance of:

(i) the class of the positive harmonic functions: if $f : \Omega_1 \to \Omega_2$ is conformal, then $h : \Omega_2 \to \mathbb{R}$ is harmonic, positive, if and only if $h \circ f : \Omega_1 \to \mathbb{R}$ is positive, harmonic;
(ii) of the hyperbolic metric.

We have just proved the invariant form of Harnack's inequality.

Theorem 60. *Let $\Omega \ne \mathbb{C}$ be simply connected, and let d_Ω be its hyperbolic metric. Let h be harmonic and positive on Ω. Then,*

$$e^{-d_h(z,w)} \le \frac{h(z)}{h(w)} \le e^{d_h(z,w)}, \tag{5.4}$$

and both estimates are best possible.

Exercise 52. *Fix the details in the proof of theorem 60.*

Often, in practice, one uses lemma 7 rather than the sharp theorem 60. The main reason is that having a good estimate for the hyperbolic distance might be not so easy. Another one is that Ω might not be conformally equivalent to the disc. The simplest way to proceed, which in some domains gives good enough estimates, is the following.

(i) For each a, we can pick $r = d_E(a, \partial\Omega)$ (the Euclidean distance!), and choose $\rho = r/2$, so the constant in lemma 7 is 9. Let z be another point of Ω, and let γ be a curve from a to z (which sometimes has to be chosen cleverly, not following the Euclidean instinct).
(ii) Cover γ with discs again having radius equal to half the distance to the boundary, and extract a finite subcover. Say that there are $N+1$ discs,

the first containing a, the last containing b. By repeated applications of the lemma, we have:

$$9^{-N} \le \frac{h(z)}{h(a)} \le 9^N.$$

The number N plays the role of the hyperbolic distance between a and z (in fact, if you are on the disc and your γ's are geodesics, it can be proved that $1/C \le \frac{N}{d_h(a,z)+1} \le C$).

This almost combinatorial procedure might give bad estimates for domains with bad behavior at some boundary point. If you work out the exercise below, you will see an example. $(*)$ Exercise to be done before writing it down.

$(*)$ State and prove Harnack's principle on limits of positive harmonic functions.

6. The non-homogeneous Laplace equation (Poisson equation)

We defined holomorphic functions by, substantially, asking that they satisfy the *homogeneous* $\bar\partial$ *equation* $\bar\partial f = 0$. In their study, however, the *fundamental solution* $\frac{1}{\pi z}$ of the corresponding *non-homogeneous equation* $\bar\partial$ popped out several times; for instance as the "least growing isolated singularity" of a holomorphic function.

What about harmonic functions? Do they have a "minimal growth" for singularities? Is it related to the non-homogeneous equation? Does it have a relation with holomorphic theory? In this subsection we provide some simple (positive) answers to these questions.

6.1. Radial harmonic functions and the minimal growth of singularities. Here we just try to understand why the logarithm is the right function to look at. If you are just interested in the solution of the Poisson equation, you can proceed to §**??**. Before we start, we make an educated guess. The fundamental solution for the $\bar\partial$ equation is $\frac{1}{\pi z}$: $\bar\partial \frac{1}{\pi z} = \delta_0$, whatever that means[1]. Now, the Laplace operator is $\Delta = 4\bar\partial\partial$, hence, to solve $\Delta G = 4\bar\partial\partial G = \delta_0$ it seems a good idea having $\partial G = \frac{1}{4\pi z}$. The naive attempt of setting $G(z) = \frac{1}{4\pi}\log z$ fails because it is not defined in a neighborhood of the origin. However, a logarithm popped out, and we expect it to play a role.[2]

We consider here *annuli* $A = \{z = re^{it} : r_1 < r < r_2\}$, with $0 < r_2 \le \infty$. A function $u : A \to \mathbb{R}$ is *radial* if $u(re^{it}) = u(r)$ only depends on r. We need the expression of the Laplace operator in polar coordinates,

$$\Delta = \partial_{rr} + \frac{\partial_r}{r} + \frac{\partial_{tt}}{r^2}. \tag{6.1}$$

LEMMA 8. *A radial function v is harmonic if and only if*

$$v(r) = \alpha \log 1/r + \beta \tag{6.2}$$

for some $\alpha, \beta \in \mathbb{R}$.

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[1]So far, it means that we can solve the non-homogeneous $\bar\partial$ equation by convolving with $\frac{1}{\pi z}$, at least in some specific cases.

[2]To do things properly in this *distributional* fashion, we should re-interpret ∂ as a gradient of a real valued function. Since we follow the classical route, we do not need doing it.

PROOF. the function $h(r) = v(r)$ (h is a function of one variable, v depends on two variables) satisfies

$$h'' + \frac{h'}{r} = 0,$$

which has the solutions (6.2). □

EXERCISE 53. *Find the radial harmonic function u on $A = \{z = re^{it} : r_1 < r < r_2\}$ ($r_2 < \infty$) such that $u(r_1) = c_1$ and $u(r_2) = c_2$.*

COROLLARY 19 (Circular averages of a harmonic function). *Let u be harmonic in the annulus $A(r_1, r_2)$. Define v on the same annulus by*

$$v(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i(t+s)})ds = v(r). \tag{6.3}$$

Then, $v(re^{it}) = \alpha \log 1/r + \beta$.

PROOF. Since $u \in C^\infty(A(r_1, r_2), \mathbb{R})$, we can take derivatives inside the integral, showing that v is harmonic. □

THEOREM 61 (Removable singularities for harmonic functions). *Let u be harmonic and real valued in $D(0, R) \setminus \{0\}$, continuous on $(clD(0, R)) \setminus \{0\}$. If*

$$\lim_{z\to 0} \frac{u(z)}{\log 1/|z|} = 0, \tag{6.4}$$

then u extends to a harmonic function on $D(0, R)$.

PROOF. Let v be the harmonic function on $D(0, R)$ which agrees with u on $\partial D(0, R)$ (v is the solution of a Dirichlet problem on a disc). Fix $\epsilon > 0$, and let

$$w(z) = u(z) - v(x) - \epsilon \log 1/|z|.$$

The function w is harmonic on $A(\delta, R)$ (for any $0 < \delta < R$), it vanishes for $|z| = R$, and by (6.4), $w(z) < 0$ for $|z| = \delta$, provided δ is small enough. By the maximum principle, $w(z) \le 0$ on $A(\delta, R)$, hence on $D(0, R) \setminus \{0\}$. As ϵ tends to zero, we have $u(z) \le v(z)$ for all z in $D(0, R)\setminus\{0\}$. The same reasoning with $u(z)-v(x)+\epsilon \log 1/|z|$ shows that $u \ge u$, hence, $u = v$ on $D(0, R) \setminus \{0\}$. The function v is the extension of u to $D(0, R)$. □

Singularities can also occur at the boundary of the domain. Are there theorems saying that there a minimal growth below which a singularity is such in a weaker form than stated is the hypothesis? The answer is positive, and it depends on the geometry of the domain and on the class of the functions one considers. These results go under the collective name of *Phragmén–Lindelöf principle*, from the name of their two independent discoverers (1908). We will consider some instances of this principle in a later chapter.

6.2. Green's formulas. Instead of looking at Green's theorem from the viewpoint of $\bar\partial$, we look at it from the viewpoint of Δ, the Laplace operator. We start from Green's theorem in the Ostrogradsky-Gauss formulation,

$$\int_\Omega \mathrm{div} V\, dxdy = \int_{\partial\Omega} V \cdot \nu d\sigma, \tag{6.5}$$

we set $V = u\nabla v$, with $u, v \in C^2(\Omega) \cap C^1(\mathrm{cl}\Omega)$, plus Riemann integrability of the second partial derivatives,

$$\int_\Omega (u\Delta v + \nabla u \cdot \nabla v) dxdy = \int_{\partial\Omega} u\frac{\partial v}{\nu} d\sigma, \tag{6.6}$$

write the same with u, v interchanged, and subtract to get rid of the gradient terms,

$$\int_\Omega (u\Delta v - v\Delta u) dxdy = \int_{\partial\Omega} \left(u\frac{\partial v}{\nu} - v\frac{\partial u}{\nu} \right) d\sigma, \tag{6.7}$$

which is *Green's formula*. The presence of the Laplace operators in the "solid" integral makes it very useful in dealing with harmonic functions.

Appendix

In this appendix I recall and discuss some results from Advanced Calculus, Topology, and Real Analysis, which are used in the lectures. Some of them might not be in the intersection of all courses of Advanced Calculus, General Topology, etcetera. When in doubt, I provide proofs. A section is devoted to the Jordan curve theorem, a result of planar topology which is crucial to the theory of conformal maps.

7. Curves and vector fields

7.1. Smooth curves and their smooth reparametrizations. A *smooth curve* in the plane (a *contour*) is a map $\gamma : [a, b] \to \mathbb{R}^2$ which is C^1 and such that $\dot{\gamma}(t) \neq 0$ for $t \in [a, b]$. The curve is a *smooth closed curve* if $\gamma(a) = \gamma(b)$ and $\dot{\gamma}(b) = \lambda\dot{\gamma}(b)$ for some $\lambda > 0$. The curve is *simple* if $\gamma(s) \neq \gamma(t)$ when $s \neq t$, with the possible exception of $\gamma(a) = \gamma(b)$. The set $\gamma([a, b])$ in $\mathbb{R}^2$ is the *image* of γ. The curve γ is *degenerate* if $a = b$ (i.e. if the image of γ is a point).

The main reason we are here interested in C^1 curves is that we can integrate vector fields over them. If $F : E \to \mathbb{R}^2$ is a continuous vector field on $E \subseteq \mathbb{R}^2$, and γ is a smooth curve in E (i.e. with image $\gamma([a, b])$ contained in E), then

$$\int_\gamma F(z) \cdot dz := \int_a^b F(\gamma(t)) \cdot \dot{\gamma}(t)dt \tag{7.1}$$

is the *integral of F along γ* (or: the *work of F along γ*). All manipulations on curves we will see are concerned with how (7.1) changes, or does not change, for various variations of γ, when the field F belongs to some specific classes.

A *smooth reparametrization* of a smooth curve γ is a smooth curve $\delta : [c, d] \to \mathbb{R}^2$, $\delta(s) = \gamma(\varphi(s))$, where $\varphi : [c, d] \to [a, b]$ is increasing, C^1, and with C^1 inverse. The curve $\delta : [c, d] \to \mathbb{R}^2$ is a *smooth reversal* if $\delta(s) = \gamma(\varphi(s))$, but with φ decreasing, C^1, and with C^1 inverse. The reason these concepts are relevant to us is the following.

PROPOSITION 20. *If δ is a smooth reparametrization of γ, then for all $F \in C(\gamma([a, b]), \mathbb{R}^2)$:*

$$\int_\delta F(z) \cdot dz = \int_\gamma F(z) \cdot dz. \tag{7.2}$$

If δ is a smooth reversal of γ, then

$$\int_\delta F(z) \cdot dz = -\int_\gamma F(z) \cdot dz. \tag{7.3}$$

Exercise 54. (i) *Prove proposition 20. Observe that the fact that φ has a C^1 inverse is not needed, and it is not even needed that $\dot\gamma \neq 0$ to define the work of F.*

(ii) *Prove that the relation "$\gamma \sim \delta$ if and only if γ is a smooth reparametrization of δ" is an equivalence relation on the smooth curves in the plane. Also prove that if $\gamma \sim \delta$ and γ is a reversal of δ, then γ is degenerate.*

(iii) *Suppose γ and δ are smooth, simple curves. Show that $\gamma \sim \delta$ if and only if, for all continuous fields F which are defined on the images of both curves, (7.2) holds.*

(iv) *Find smooth curves $\gamma \nsim \delta$ (not simple ones) for which (7.2) holds.*

Reparametrizations and reversals put "arrows" on the image of a curve. The curve is a kinematic object (a trajectory in space, which is function of "time"), its image is a geometric object (a figure). The equivalence relation in (ii) defines a geometric object at a higher structural level, in which we have a figure from a special class and an orientation for it.

We will later consider the special class of the *exact* (or *conservative*) fields. In that case, the modifications on the curves (or on combinations of curves) which preserve the work are much richer, and they include *homotopy* and *homology*. The analysis of fields was, in fact, the starting point for the development of algebraic topology, in the second part of the XIX century. See the first few pages in History of homological algebra by Charles A. Weibel.

Closed, smooth curves have a richer family of reparametrizations. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a closed, smooth curve, and let $c \in (a,b)$. Define the new curve $\gamma_c[c+a-b,c] \to \mathbb{R}^2$ as

$$\gamma_c(t) = \begin{cases} \gamma(t) \text{ if } a \le t \le c, \\ \gamma(t+b-a) \text{ if } c+a-b \le t \le a. \end{cases} \tag{7.4}$$

We say that δ is a *smooth circle reparametrization* of a smooth, closed curve γ if δ is a reparametrization of γ_c for some $c \in [a,b]$; and δ is a *smooth circle reversal* of γ if δ is a smooth reversal of some γ_c.

In fact, it is more natural to view smooth, closed curves as C^1 maps $\gamma : \mathbb{T} \to \mathbb{R}^2$, where $\mathbb{T} = \{(\cos t, \sin t) = e^{it} : t \in [0, 2\pi]\}$ is the 1-dimensional *torus* (i.e. the unit circle). To formalize the definition, we might view $\mathbb{T}$ as a 1-dimensional manifolds, which carries a differentiable structure.

Proposition 21. *If δ is a smooth circle reparametrization of a closed, smooth γ, then for all $F \in C(\gamma([a,b]), \mathbb{R}^2)$:*

$$\int_\delta F(z) \cdot dz = \int_\gamma F(z) \cdot dz. \tag{7.5}$$

If δ is a smooth, circle reversal of a closed, smooth γ, then

$$\int_\delta F(z) \cdot dz = -\int_\gamma F(z) \cdot dz. \tag{7.6}$$

Exercise 55. *Rephrase and prove the statements of 54 in the world of the closed curves.*

Equation (7.6) generalizes the well known relation $\int_a^b h(x)dx = -\int_b^a h(x)dx$ which holds for definite integral in one variable. It is especially interesting, because it encodes a *cancellation property* hidden in integrals of vector fields. Following γ

by δ we have a non-degenerate curve, which is nonetheless transparent to vector fields, from the viewpoint of line integrals, and can be ignored in calculations.

In the case of the closed curves, orientation and order when an orientation is fixed behave like the corresponding notions on a circle. We have two orientations: "clock-wise" and "anti-clockwise"; and the phrase "in the given orientation we find the point C in the arc which goes from A to B" makes sense.

7.2. Differential forms. We will often use the language of differential forms, which is very convenient in calculations, and highlights the invariance of several properties under change of variables. The definition is rigorous, but naive. If you want to have more general and intrinsic way to define them, see [**Rudin**] or, even better, [**Flanders**].

(0) A 0-form on an open subset Ω of the plane is simply a function $f(x, y)$.

(1) A 1-form on Ω is an object of the form $\psi = P(x, y)dy + Q(x, y)dy$, where P, Q are functions of $z = (x, y)$. As far as we are concerned here, you might think of ψ as the vector field (P, Q) (rather, as its *dual* with respect to the Euclidean metric, but here we are at a much lower level of sophistication).

(2) A 2-form on Ω is is an object of the form $\Psi = m(x, y)dx \wedge dy$, where $\wedge$ is linear in both components and anti-symmetric, $df \wedge dg = -dg \wedge df$.

(d) We introduce the linear differential operator d which maps 0-forms to 1-forms,
$$df = \partial_x f dx + \partial_y f dy,$$
and 1-forms to 2-forms, which acts according to the rule:
$$d(f dg) = df \wedge dg.$$
In particular, $ddf = 0$. Hence,
$$\begin{aligned} d(Pdx + Qdy) &= dP \wedge dx + dQ \wedge dy \\ &= (P_x dx + P_y dy) \wedge dx + (Q_x dx + Q_y dy) \wedge dy \\ &= P_y dy \wedge dx + Q_x dx \wedge dy = (Q_x - P_y)dx \wedge dy. \end{aligned}$$

($\int$) 1-forms can be integrated on curves $z = \gamma(t)$, $t \in [a, b]$,
$$\int_\gamma Pdx + Qdy = \int_a^b (P(z(t))\dot{x}(t) + Q(z(t))\dot{y}(t))dt,$$
and 2-forms can be integrated over regions of the plane,
$$\int_A m(x, y)dx \wedge dy = \int_A M(x, y)dxdy.$$

8. Green-Ostrogradsky-Gauss-Stokes

The goal of this section is stating and proving a version of *Green's theorem* in the plane. It is central to Cauchy's line integral approach to holomorphic functions, and in fact Cauchy himself provided his own proof while developing an important chapter of holomorphic function theory. The formula, in *Green's formulation*, is the following.

$$\int_{\partial\Omega} Pdx + Qdy = \int_\Omega (\partial_x Q - \partial_y P)dxdy, \tag{8.1}$$

where

(i) Ω is open, bounded, and connected (a *domain*) in the plane;

(ii) $\partial\Omega$ is the union of images of C^1, simple, closed curves (rectifiable suffices), having Ω on their left (*positively oriented*, with respect to Ω);
(iii) $P, Q \in C^1(\Omega) \cap C(\mathrm{cl}\Omega)$, and $\partial_y P$, $\partial_x Q$ are Riemann integrable on Ω (Riemann integrability of $\partial_x Q - \partial_y P$ suffices.)

On the left hand side of (8.1), the integral is defined as follows. If $z = \gamma = (x, y) : [a, b] \to \mathbb{R}^2$ is one of the positively oriented curves on $\partial\Omega$, and $z = (x, y)$,

$$\int_\gamma P(z)dx + Q(z)dy = \int_a^b (P(z(t)\dot{x}(t) + Q(z(t))\dot{y}(t))dt. \tag{8.2}$$

The most visual version of the above statement, however, is the *Ostrogradsky-Gauss divergence theorem*,

$$\int_{\partial\Omega} F \cdot \nu d\sigma = \int_\Omega \mathrm{div} F dm, \tag{8.3}$$

where

(i) ν is the *unit vector normal to* $\partial\Omega$, *pointing outside* Ω;
(ii) $d\sigma$ is the *length element* on $\partial\Omega$;
(iii) $\mathrm{div}F$ is the *divergence* of F;
(iv) dm is *area (Lebesgue) measure* restricted to Ω.

This is also the version which is easiest to read in terms of classical physics: $\mathrm{div}F$ measures the balance of the *flux* of F through $\partial\Omega$. If F is the velocity field of a fluid, for instance, $A = \int_{\partial\Omega} F \cdot \nu d\sigma$ measures the volume of the fluid which exits $\partial\Omega$ (if $A \geq 0$), or gets entrapped in Ω (if $A \leq 0$) in a unit of time.

The modern, geometric way to write it, however, is *Stokes' theorem*, which is the elegant:

$$\int_{\partial\Omega} \omega = \int_\Omega d\omega, \tag{8.4}$$

where

(i) ω is a 1-*form*;
(ii) $d\omega$, its *differential*, is a 2-*form*.

The identity might be read as a very general fundamental theorem of calculus (which might be written in this form). More important, the way it is written suggests a number of extensions and generalizations, in Euclidean spaces and on manifolds.

This material is usually covered in Advanced Calculus, or in a class on Calculus on Surfaces. Since it is central to our line of argument, we provide here precise statements and definitions, and complete proofs. We will also develop some theory related to the work of a field along a curve, of which the left hand side of (8.3) is a special case, which will play a prominent role later on.

8.1. The simplest proof of Green's theorem. Below we provide two proofs of Green's theorem: one which is classical, and a more modern (1963) one which works for more general boundaries. If you have never seen the topic before, however, you might want to read the simplest, possibly the first, proof, without which the other proofs might look like magics. The simple proof works well for special domains, and more general versions can be obtained by gluing them together, or, viceversa, decomposing the domain you are dealing with in a number of special domains. This is enough for most applications to classical physics and engineering.

THEOREM 62. *Let* $\varphi : [a,b] \to \mathbb{R}$ *be* C^1 *with* $\varphi'(x) \neq 0$ *in* $[a,b]$*,* $\varphi(x) > c$ *on* $[a,b]$*, and let*

$$\Omega = \{(x,y) : x \in (a,b), c < y < \varphi(x)\}.$$

Let $P, Q \in C^1(cl\Omega)$*. Then,*

$$\int_{\partial\Omega}(Pdx + Qdy) = \int_{\Omega}(\partial_x Q - \partial_y P)dxdy. \tag{8.5}$$

PROOF. Let $\gamma_1(x) = (x,c)$ $(x \in [a,b])$, $\gamma_2(y) = (b,y)$ $(y \in [c,\varphi(b)])$, $\gamma_3^{-1}(x) = (x,\varphi(x))$ $(x \in [a,b])$, $\gamma_4^{-1}(y) = (a,y)$ $(y \in [c,\varphi(a)])$, so that $\partial\Omega = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. We also have $\dot\gamma_1(x) = (1,0)$, $\dot\gamma_y(y) = (0,1)$, $\dot\gamma_3(x) = (-1,-\varphi'(x))$, $\dot\gamma_4(y) = (0,-1)$. For the P terms everything goes smoothly:

$$\begin{aligned}
\int_\Omega \partial_y P dxdy &= \int_a^b \left(\int_c^{\varphi(x)} \partial_y P(x,y)dy\right) dx \\
&= \int_a^b \left(P(x,\varphi(x)) - P(x,c)\right) dx \\
&= -\int_{\partial\Omega} Pdx.
\end{aligned}$$

For the Q term, consider $\psi(y) = \varphi^{-1}(y)$, which maps the interval having endpoints $\varphi(a), \varphi(b)$ onto $[a,b]$. Then,

$$\begin{aligned}
\int_\Omega \partial_x Q dxdy &= \int_c^{\varphi(a)} \left(\int_a^b \partial_x Q(x,y)dx\right) dy + \int_{\varphi(a)}^{\varphi(b)} \left(\int_{\psi(y)}^b \partial_x Q(x,y)dx\right) dy \\
&= \int_c^{\varphi(a)} \left(Q(b,y) - Q(a,y)\right) dy + \int_{\varphi(a)}^{\varphi(b)} \left(Q(b,y) - Q(\psi(y),y)\right) dy \\
&= \int_c^{\varphi(b)} Q(b,y)dy - \int_c^{\varphi(a)} Q(a,y)dy - \int_{\varphi(a)}^{\varphi(b)} Q(\psi(y),y)dy \\
&= \int_c^{\varphi(b)} Q(b,y)dy - \int_c^{\varphi(a)} Q(a,y)dy - \int_a^b Q(x,\varphi(x))\varphi'(x)dx \\
&= \int_{\partial\Omega} Qdy.
\end{aligned}$$

□

The relation holds if we exchange x and y, and if the graph lies below $y = c$ instead of above it. When two such regions are glued together, the line integrals on the common part of the boundary cancel out, hence the statement hold for the larger region. This is how the theorem is used in many applications to concrete cases. A weakness of it is that a C^1 curve is not locally the graph of an increasing/decreasing function of x, or y. We might remedy by adding to the tiles rotations of the special domains. However, this does not seem to be the shortest way to prove a more general statement.

EXERCISE 56. *Prove that* (8.1) *is invariant under smooth changes of coordinates (it is, that is, a theorem in differential topology, rather than in differential geometry). Explicitly, let* $z = (x,y) = \Phi(u,v)$*, and show that* (8.1) *transforms in a copy of itself in the new coordinates. For which new* P *and which new* Q*?*

You might assume that Φ *maps* $\mathbb{R}^2$ *onto itself diffeomorphically (*C^1 *with* C^1 *inverse).*

8.2. Green's theorem for C^1 boundaries. In this subsection, we give a proof of the following version of Green's theorem. Almost all the material we cover, however, only requires the easy to prove, local version of the theorem. This global version is used to produce some global statements about holomorphic functions, which are not anyhow central in the development of the the theory during this course.

THEOREM 63 (Green's theorem in regions with C^1 boundary). *Let Ω be a bounded, connected, open subset of $\mathbb{R}^2$, having as boundary $\gamma_1(I_1) \cup \cdots \cup \gamma_k(I_k)$, where each γ_i is a closed, regular, simple curve $\gamma_i : I_i = [a_i, b_i] \to \mathbb{R}^2$, and the union of their traces is disjoint. Let $P, Q : cl\Omega \to \mathbb{R}$, $P, Q \in C^1(\Omega) \cap C(cl\Omega)$, and $\partial_x Q$, $\partial_y P$ are Riemann integrable (hence, bounded) in Ω. The curves are parametrized to have Ω on their left.*

Then,

$$\int_{\partial\Omega}(Pdx + Qdy) = \int_{\Omega}(\partial_x Q - \partial_y P)dxdy. \tag{8.6}$$

This version is rather general, but it does not cover the often encountered case of domains with corners or cusps. As a matter of fact, in the course of the proof we will cover special domains with corners, and by gluing them one can prove Green's theorem for many domains with piecewise regular boundary. In the next subsection, we prove Green's formula for rectifiable boundaries, which in particular covers the case of piecewise smooth boundaries. Alternatively, one might approximate a domain with piecewise regular boundary by inner domains with smooth boundary.

8.2.1. *Green's theorem: the basic tile.* We give here a more general version of theorem 62, where φ is not supposed to be strictly increasing.

THEOREM 64. *Let $\varphi : [a, b] \to \mathbb{R}$ be C^1, $\varphi(x) > c$ on $[a, b]$, and let*

$$\Omega = \{(x, y) : x \in (a, b), c < y < \varphi(x)\}.$$

Let $P, Q \in C^1(\Omega) \cap (cl\Omega)$, and suppose that $\partial_x Q$, $\partial_y P$ are Riemann integrable on Ω. Then,

$$\int_{\partial\Omega}(Pdx + Qdy) = \int_{\Omega}(\partial_x Q - \partial_y P)dxdy. \tag{8.7}$$

We prove the theorem for $P, Q \in C^1(\mathrm{cl}\Omega)$, and the general follows by approximating Ω by domains Ω_ϵ with closure contained in D. More precisely, for $\epsilon > 0$, let

$$\Omega_\epsilon = \{(x, y) : x \in (a + \epsilon, b - \epsilon), c + \epsilon < y < \varphi(x) - \epsilon\}. \tag{8.8}$$

If we prove (8.7) for this domain, the general case follows by taking the limit as $\epsilon \to 0$.

LEMMA 9 (Derivative under integral sign). *Let $v \in C^1([a, b])$, $v \geq 0$, and $f \in C^1(clD)$, where $D = \{(x, y) : a < x < b, 0 < y < \varphi(x)\}$. Define*

$$\psi(x) = \int_0^{\varphi(x)} f(x, y)dy. \tag{8.9}$$

Then, for $x \in (a, b)$,

$$\psi'(x) = \int_0^{\varphi(x)} \partial_x f(x, y)dy + f(x, \varphi(x))\varphi'(x). \tag{8.10}$$

Proof. Let $G(x,z) = \int_0^z f(x,y)dy$, $G : \mathrm{cl}D \to \mathbb{R}$. Clearly $\partial_z G(x,z) = f(x,z)$. For the partial derivative with respect to x, using Lagrange's mean value with $\theta = \theta(x,y,h)$

$$\begin{aligned} \frac{G(x+h,z)-G(x,z)}{h} - \int_0^z \partial_x f(x,y)dy &= \int_0^z \left(\frac{f(x+h,y)-f(x,y)}{h} - \partial_x f(x,y)\right) dy \\ &= \int_0^z (\partial_x f(x+\theta h, y) - \partial_x f(x,y))dy, \end{aligned}$$

which tends to zero as $h \to 0$, since $\partial_x f$ is uniformly continuous. Thus,

$$\partial_x G(x,z) = \int_0^z \partial_x f(x,y)dy. \tag{8.11}$$

Since $\psi(x) = G(x,\varphi(x))$, $\psi'(x) = \partial_x(x,\varphi(x)) + \partial_z G(x,\varphi(x))\varphi'(x)$ is the expression in 8.10. □

Proof of theorem 64. . The term containing P is not problematic,

$$\begin{aligned} \int_a^b \left(\int_c^{\varphi(x)} \partial_y P(x,y)dy\right) dx &= \int_a^b [P(x,\varphi(x)) - P(x,c)] \\ &= -\int_{\partial D} P(x,y)dx. \end{aligned}$$

For the other term, we use lemma 8.9:

$$\begin{aligned} \int_a^b \left(\int_c^{\varphi(x)} \partial_x Q(x,y)dy\right) dx &= \int_a^b \left(\frac{d}{dx}\int_c^{\varphi(x)} Q(x,y)dy - Q(x,\varphi(x))\varphi'(x)\right) dx \\ &= \int_c^{\varphi(b)} Q(b,y)dy - \int_c^{\varphi(a)} Q(a,y)dy \\ &\quad - \int_a^b Q(x,\varphi(x))\varphi'(x)dx \\ &= \int_{\partial D} Q(x,y)dy. \end{aligned}$$

□

The general case where the domain has piecewise C^1 boundary can be deduced by decomposing it into domains of the special form of theorem 64.

8.2.2. *Smooth curves are locally graphs.* There is nothing dramatically "fractal" about smooth curves.

Theorem 65. *Let $\gamma : [a,b] \to \Omega \subseteq \mathbb{R}^2$ be a smooth, simple, non-degenerate curve, lying of $\partial\Omega$, where $\partial\Omega$ is the union of finitely many, disjoint curves of the same class. Let $a < t_0 < b$, and set $\gamma(t_0) = (x_0, y_0)$. Then, either there exists a rectangular neighborhood $\boldsymbol{R} = (x_0 - \delta, x_0 + \delta) \times (y_0 - \epsilon, y_0 + \epsilon)$ in Ω, and a function $\varphi \in C^1((x_0 - \delta, x_0 + \delta), (y_0 - \epsilon/2, y_0 + \epsilon/2))$ such that $\gamma(t) \in \boldsymbol{R}$ if and only if $y = \varphi(x)$; or the same property holds with x and y interchanged.*

Moreover, $\boldsymbol{Q} \setminus \gamma([a,b]) = \boldsymbol{R}_- \cup \boldsymbol{R}_+$, where $\boldsymbol{R}_+$ and $\boldsymbol{R}_-$ are the super/sub-graphs of φ in $\boldsymbol{R}$, which are open, non-empty, and connected.

The parameters δ, ϵ can be taken to be independent of the point (x_0, y_0) on the curve.

The statement provides a rigorous meaning to the informal expression *the domain* Ω *is on the left of* γ. If the portion of γ inside $\mathbf{R}$ is, after reparametrization, $x \mapsto (x, \varphi(x))$ and $\Omega \cap \mathbf{R} = \mathbf{R}_+$, then Ω is on the left of γ in $\mathbf{R}$. If the same portion of γ is instead a reversal of the graph of φ, and $\Omega \cap \mathbf{R} = \mathbf{R}_+$, then Ω is on the right of γ in $\mathbf{R}$. When $\Omega \cap \mathbf{R} = \mathbf{R}_-$, left and right are reversed. In fact, the notion of left/right can be given independently of the rectangular neighborhood.

This notion of right and left does not depend on the rectangular neighborhood. In the intersection of two such neighborhoods, the points are on the right or on the left of γ in both.

Corollary 20. *Let* $\gamma : [a, b] \to \Omega \subseteq \mathbb{R}^2$ *be a smooth, simple, non-degenerate curve, lying of* $\partial\Omega$, *where* $\partial\Omega$ *is the union of finitely many, disjoint curves of the same class. Write* $z(t) = \gamma(t) = (x(t), y(t))$ *as usual.*

Then, Ω *lies on the left of* γ *if and only if for one (equivalently: for all) points* $z_0 = \gamma(t_0)$ *on the curve,*

$$\nu(t) := z_0 + t(\dot{y}(t_0), -\dot{x}(t_0)) \in \mathbb{C} \setminus (cl\Omega) \text{ for } 0 < t < \epsilon,$$

for some positive ϵ.

Exercise 57. *Prove 20.*

Proof of theorem 65. Let $\gamma(t) = (x(t), y(t))$, and $\nu = (-\dot{y}(0), \dot{x}(0)) \neq (0, 0)$, by definition of smooth curve. Consider the map

$$F(t, s) = \gamma(t) + s\nu,$$

$F : (a, b) \times (-\delta_0, \delta_0) \to \Omega$, provided $\delta_0 > 0$ is small enough (by compactness of $\gamma([a, b])$). The function F is clearly C^1. Also,

$$\det JF(s, t_0) = \det[\dot{\gamma}(t_0)|\nu] = \dot{x}(t_0)^2 + \dot{y}(t_0)^2 > 0,$$

hence $JF(t, s)$ is non-singular for $|t - t_0| \leq \delta_1$ for some $\delta_1 > 0$. (By uniform continuity, δ_1 can be taken to be independent from t_0). By the inverse mapping theorem, there exist $\eta > 0$ (again, independent of t_0) and an open neighborhood V of $(t_0, 0)$ in $(a, b) \times (-\delta, \delta)$ such that F maps V onto the open set $F(V)$ diffeomorphically (injective, with smooth inverse $G : F(V) \to V$).

Consider the inverse map $G(x, y) = (h(x, y), k(x, y)) = (t, s)$, and consider the point $\gamma(t_0) = F(t_0, 0)$. If $\nabla k(\gamma(t_0)) = 0$, $JG(\gamma(t_0)) = (JF(t_0, 0))^{-1}$ would be singular, which is absurd. Suppose that $\partial_y k(\gamma(t_0)) \neq 0$. By the implicit function theorem, there exists a neighborhood W of $\gamma(t_0) = (x_0, y_0)$ in $F(V)$, an interval $(x_0 - \epsilon, x_0 + \epsilon)$, and a C^1 function $\varphi : (x_0 - \epsilon, x_0 + \epsilon) \to \mathbb{R}$ such that:

(i) $h(x, y) = 0$ with $(x, y) \in W$ if and only if $y = \varphi(x)$ with $x \in (x_0 - \epsilon, x_0 + \epsilon)$;
(ii) in particular, $\varphi(x_0) = y_0$.

Since $h(x, y) = 0$ if and only if (x, y) lies on the portion of the image of γ which lies in $F(V) \supseteq W$, we have that $\gamma([a, b]) \cap W$ is the graph of φ. By making W smaller, we can take it to be a rectangle

$$W = (x_0 - \delta, x_0 + \delta) \times (y_0 - \epsilon, y_0 + \epsilon) \tag{8.12}$$

for suitable $\epsilon, \delta > 0$, with $|\varphi(x) - y_0| < \epsilon/2$ if $|x - x_0| < \delta$. Its preimage under F is $F^{-1}(W)$, which is an open set in the (t, s)-plane, containing the point $(t_0, 0)$. $\square$

8.2.3. *Proof of theorem 63.* In theorem 65 we assumed γ is C^1. The proof of theorem 63 will be given assuming all boundary curves are smooth. After the proof, we suggest how the proof can be modified to hold for the more general case.

We actually provide two proofs for the smooth case. The first one is shorter, but it makes use of partitions of unity. If you haven't see those, a different proof is provided below.

LEMMA 10 (Partition of unity). *Let K be compact in $\mathbb{R}^d$, and $\{U_j\}_{j=1}^n$ be a finite, open cover of K. Then, there exist functions $\eta_j \in C_c^\infty(U_j)$ such that $\eta_1 + \cdots + \eta_n = 1$ on K.*

PROOF OF THEOREM 63. For each point ζ in $\partial\Omega$, consider an open rectangle D_ζ centered at it such that $\gamma \cap D_\zeta$ is the graph of a function which is defined on the projection of the rectangle on one of the axis, as in theorem 65. Let $D_1, \dots, D_n$ be a covering of γ by finitely many such rectangles. The set $H = \Omega \setminus (\cup_{j=1}^n U_j)$ is compact in Ω, then there exists D_0 open, containing H, and with closure contained in Ω. Consider a partition of unity $\{\eta_j\}_{j=0}^n$, subordinated to the covering $\{U_j\}_{j=0}^n$ of $\mathrm{cl}\Omega$.

On the left of (8.6) we have

$$\int_{\partial\Omega} Pdx + Qdy = \sum_{j=0}^n \int_{\partial\Omega} (\eta_j Pdx + \eta_j Qdy),$$

while on the right we have

$$\begin{aligned}\int_\Omega (\partial_x Q - \partial_y P)dxdy &= \int_\Omega \sum_{j=0}^n \eta_j(\partial_x Q - \partial_y P)dxdy \\ &= \sum_{j=0}^n \int_\Omega [\partial_x(\eta_j Q) - \partial_y(\eta_j P)]dxdy - \sum_{j=0}^n \int_\Omega [\partial_x\eta_j Q - \partial_y\eta_j P]dxdy \\ &= \sum_{j=0}^n \int_\Omega [\partial_x(\eta_j Q) - \partial_y(\eta_j P)]dxdy,\end{aligned}$$

because $\sum_{j=0}^n \partial_x\eta_j = \partial_x(\sum_{j=0}^n \eta_j) = 0 = \sum_{j=0}^n \partial_y\eta_j$ on $\mathrm{cl}\Omega$. It suffices, then, to prove that

$$\int_{\partial\Omega} (\eta_j Pdx + \eta_j Qdy) = \int_\Omega [\partial_x(\eta_j Q) - \partial_y(\eta_j P)]dxdy, \tag{8.13}$$

i.e. that the theorem holds if P and Q are supported in one of the D_j's. We can consider the Q term, the calculations for P being identical.

The term with $j = 0$ is not problematic. Let $R > 0$ be so large that the projection of Ω on the y-axis lies in $[-R, R]$, and for $-R \le y \le R$, let $a(y)$ (resp. $b(y)$) be the smallest (resp. largest) abscissa of points in D_0 having ordinate y.

$$\begin{aligned}\int_\Omega \partial_x(\eta_0 Q)dxdy &= \int_{-R}^R \left(\int_{a(y)}^{b(y)} \partial_x(\eta_0 Q)dx\right) dy \\ &= \int_{-R}^R (\eta_0(y, b(y))Q(y, b(y)) - \eta_0(y, a(y))Q(y, a(y)))\, dy \\ &= 0 \\ &= \int_{\partial\Omega} (\eta_0 Pdx + \eta_0 Qdy),\end{aligned}$$

where the last equality holds because η_0 vanishes on $\partial\Omega$.

For the terms with $j \geq 1$ we use theorem 65:

$$\begin{aligned}\int_\Omega \partial_x(\eta_j Q)dxdy &= \int_{D_j\cap\Omega} \partial_x(\eta_j Q)dxdy \\ &= \int_{\partial(D_j\cap\Omega)} (\eta_j Pdx + \eta_j Qdy) \\ &= \int_{D_j\cap(\partial\Omega)} (\eta_j Pdx + \eta_j Qdy) \\ &= \int_{\partial\Omega} (\eta_j Pdx + \eta_j Qdy)\end{aligned}$$

where the third equality holds because η_j vanishes on the three segment of which $\partial(D_j \cap \Omega) \cap \Omega$ is composed. □

8.3. Invariance under diffeomorphisms. One of the nice features of Green's formula, which shows that it belongs to differential topology, is its invariance under diffeomorphisms. We are not concerned here with the weakest regularity assumtions.

Suppose A, B are domains in $\mathbb{R}^2$, and that $z = \Phi(w)$ is a C^2 diffeomorphism (a bijection with non-singualr Jacobian), $\Phi : A \to B$. Let $\Omega \subset \mathrm{cl}\Omega \subset B$ be a domain with regular C^1 boundary.

THEOREM 66. *If Green's theorem holds in $\Phi^{-1}(\Omega)$, then it holds in Ω.*

In theorem 62 we saw that Green's theorem can used on a special class of domains, and we already know that we extend that class by gluing. With lemma **??**, we can also add to the list of the good domains the diffeomorphic images of good domains (where the diffeomorphism has to be C^2 for our proof to work). In the proof we see how the two sides of Green's theorem transform under Φ.

PROOF. We use the variables $z = (x, y)$ and $w = (u, v)$. Let P, Q be C^1 on $\mathrm{cl}\Omega$. The line integral in Green's theorem transforms as

$$\begin{aligned}\int_{\partial\Omega} Pdx + Qdy &= \int_{\Phi^{-1}(\partial\Omega)} P(x_u du + x_v dv) + Q(y_u du + y_v dv) \\ &= \int_{\Phi^{-1}(\partial\Omega)} (Px_u + Qy_u)du + (Px_v + Qy_v)dv \\ &= \int_{\Phi^{-1}(\partial\Omega)} Rdu + Sdv,\end{aligned}$$

where

$$R(u,v) = Px_u + Qy_u = P(\Phi(u,v))\partial_u x(u,v) + Q(\Phi(u,v))\partial_u y(u,v),$$

$$S(u,v) = Px_v + Qy_v = P(\Phi(u,v))\partial_v x(u,v) + Q(\Phi(u,v))\partial_v y(u,v). \tag{8.14}$$

The area integral transforms as

$$\begin{aligned}\int_\Omega (Q_x - P_y)dx \wedge dy &= \int_{\Phi^{-1}(\Omega)} (Q_x - P_y)(x_u du + x_v dv) \wedge (y_u du + y_v dv) \\ &= \int_{\Phi^{-1}(\Omega)} (Q_x - P_y)(x_u y_v - x_v y_u)dudv.\end{aligned}$$

To obtain Green's theorem we just need to verify that

$$(Q_x - P_y)(x_u y_v - x_v y_u) = S_u - R_v,$$

which you are invited to verify yourself. Here is however the calculation:

$$\begin{aligned} S_u - R_v &= (P_x x_u + P_y y_u)x_v + (Q_x x_u + Q_y y_u)y_v - (P_x x_v + P_y y_v)x_u \\ &\quad -(Q_x x_v + Q_y y_v)y_u + P x_{vu} + Q y_{vu} - P x_{uv} - Q y_{uv} \\ &= (Q_x - P_y)(x_u y_v - x_v y_u), \end{aligned}$$

as wished. □

9. Irrotational and conservative forms

Definition 8. *A differential* 1*-form* $\omega = P(x,y)dx + Q(x,y)dy$ *with coefficients* P, Q *which are continuous in a domain* Ω *is* **conservative** *(or* **exact***) in an open subset* A *of* Ω *if it has a* **potential***: a function* $\varphi \in C^1(A, \mathbb{R})$ *such that*

$$P = \partial_x \varphi, \ Q = \partial_y \varphi. \tag{9.1}$$

We say that ω *is* **irrotational** *(or* **closed***) if*

$$\partial_y P = \partial_x Q. \tag{9.2}$$

In the language of 1-forms, the conservative $Pdx + Qdy = \partial_x \varphi dx + \partial_y \varphi dy = d\varphi$ is the differential of φ. By Schwarz lemma on mixed derivatives, if P, Q are C^1, then *conservative* $\implies$ *irrotational.* In the language of forms, the 1-form ω is irrotational if $d\omega = 0$, and Schwarz lemma says

$$\text{if } \omega = d\varphi, \text{ then } d\omega = 0,$$

which reduces to

$$dd\varphi = 0. \tag{9.3}$$

The following items are common to all courses of Advanced Analysis, and we will not prove them.

- (i) The following assertion are equivalent for a 1-form ω with continuous coefficients in a domain A:
  - (a) ω is irrotational;
  - (b) $\int_\gamma \omega = 0$ for all regular, closed curve in A;
  - (c) $\int_\alpha \omega = \int_\beta \omega$ for all regular curves in A having the same initial and final endpoint.
- (ii) With A and ω as in (i), a potential φ for ω is given by

  $$\varphi(z) = \int_{\gamma_z} \omega,$$

  where γ_z is a regular curve starting at some fixed $a \in A$ and ending at z. Moreover, all other potentials have the form $\varphi + k$ for some k in A.

The thesis of the Volterra-Poincaré theorem can be read as

$$\text{if } d\omega = 0, \text{ then } \omega = d\varphi \text{ for some } \varphi.$$

Definition 9. *An open set* Ω *has the* **Volterra-Poincaré property** *if all irrotational* 1*-forms on* Ω *are conservative.*

Another way to phrase the definition is that, for all regular, closed curves γ in Ω and all irrotational 1-forms ω on Ω,

$$\int_\gamma \omega = 0, \tag{9.4}$$

A third way to state the property is:

$$\ker d = \mathrm{Ran} d, \tag{9.5}$$

where Ran$d$ is the range of the differential operator d, the differential on the left of (9.5) acts on 1-forms, while that on the right acts on 0-forms (functions).

This is a temporary definition, because we will see during the course that the following global properties are in fact equivalent:

(i) Ω has the Volterra-Poincaré property;
(ii) Ω is *simply connected* (i.e. each closed curve in Ω is *homotopic* to a constant);
(iii) $\mathbb{C}_* \setminus \Omega$ is connected, where $\mathbb{C}_*$ is the one point compactification of $\mathbb{C}$;
(iv) for each closed curve γ in Ω and a in $\mathbb{C} \setminus \Omega$, $n(\gamma, a) = 0$, where $n(\gamma, a)$ is the *index* of γ with respect to a.

The following is a basic result of Advanced Analysis.

THEOREM 67 (Local Volterra-Poincaré theorem). *If Ω is convex, then it has the Volterra-Poincaré property.*

Theorem 67 and invariance under smooth changes of coordinates suffices for most practical purposes.

PROPOSITION 22 (The Volterra-Poincaré property is invariant under diffeomorphisms). *Let Ω be open in the plane with the Volterra-Poincaré property, and let $\Phi : A \to \Omega$ be a C^1 diffeomorphism. Then, A has the Volterra-Poincaré property.*

PROOF. We use coordinates $z = (x, y)$ in Ω and $w = (u, v)$ in A. Let $\gamma : [a, b] \to \Phi^{-1}(\Omega)$ be a closed, regular curve. Then, $\Phi \circ \gamma$ is a closed, regular curve in Ω, hence,

$$\int_{\Phi\circ\gamma} P(z)dx + Q(z)dy = 0 \tag{9.6}$$

whenever P, Q are C^1 in Ω. Let $C, D \in C^1(A)$ arbitrary. If we find $P, Q \in C^1(\Omega)$ such that:

$$\int_{\Phi\circ\gamma} P(z)dx + Q(z)dy = \int_\gamma C(w)du + D(w)dv, \tag{9.7}$$

we have proved that A has the Volterra-Poincaré property. Unraveling definitions, (9.7) holds if

$$\begin{aligned} C(w)du + D(w)dv &= P(z)dx + Q(z)dy \\ &= P(z(w))(x_u du + x_v dv) + Q(z(w))(y_u du + y_v dv) \\ &= [P(z(w))x_u + Q(z(w))y_u]du + [P(z(w))x_v + Q(z(w))y_v]dv, \end{aligned} \tag{9.8}$$

and we can solve for P, Q because $J\Phi = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$ is a non-singular matrix. □

Actually, the proof says more than the statement: it explicits how 1-forms and their line integrals, and their differentials and the corresponding area integral, transform under smooth maps. It is worth the effort of making this into a statement.

PROPOSITION 23. *Let $\Phi : A \to \Omega$ be a C^1-map*

EXERCISE 58. *Suppose that Φ is a diffeomorphism from A to Ω which extends to a C^1 map on clA, and that C and D are in $C^1_c(clA)$. Show that if (C,D) and (P,Q) are related as in* (9.8)*, then*

$$\int_A (D_u - C_v)du \wedge dv = \int_\Omega (Q_x - P_y)dx \wedge dy.$$

[The wedge product takes into account possible changes in orientation].

10. The Jordan curve theorem (work in progress)

In this section we are going to prove the following.

THEOREM 68 (Jordan curve theorem). *Let $\gamma : \mathbb{T} \to \mathbb{R}^2$ be a homeomorphism (i.e. γ is a simple, closed curve in the plane). Then, $\mathbb{R}^2 \setminus \gamma(\mathbb{T})$ has exactly two connected components: Ω_∞ (unbounded) and Ω (bounded). Moreover, $\partial\Omega_\infty = \partial\Omega = \gamma(\mathbb{T})$.*

The statement is not really surprising, but all of its known proofs are either long and tricky, or they rely on nontrivial results in algebraic topology. The fact is that the homeomorphic image of a circle can be more intricate than a closed line drawn on paper. In drawings, for instance, you can not really picture self-similarity, which is a feature of many simple, closed curves. A survey of mathematicians that either used the Jordan curve theorem or one of its consequences in their research, or that used it without proof in a course of Complex Analysis, would probably show that many of them never saw a complete proof of it. In this section, you have a chance to outsmart them.

The proof I give is from The Jordan Curve Theorem Via the Brouwer Fixed Point Theorem, by Ryuji Maehara (The American Mathematical Monthly Vol. 91, No. 10, 1984, pp. 641-643. For the proof of the Brouwer fixed point theorem I follow the "game theory proof" in The Game of Hex and the Brouwer Fixed-Point Theorem, by David Gale (The American Mathematical Monthly Vol. 86, No. 10, 1979, pp. 818-827).

Another proof I like very much, which ends in the stronger *Jordan-Schönflies theorem*, is the graph-theoretic one in The Jordan-Schönflies Theorem and the Classification of Surfaces, by Carsten Thomassen (The American Mathematical Monthly Vol. 99 no. 2, 1992, pp. 116-130). The original proof by Jordan is in his 1887 lectures. Some thought he had left some gaps in the line of argument, and more proofs were provided. The current view is that his proof was solid. See the Wikipedia page Jordan curve theorem (with link to Jordan's work), and the essay Jordan's Proof of the Jordan Curve Theorem by Thomas C. Hales (Studies in logic, grammar and rhetoric 10 (23) 2007), for an overview of the theorem's history. The statement extending the Jordan curve theorem to higher dimension holds true (it's called the *Jordan-Brouwer separation theorem*, independently proved by Brouwer and Lebesgue in 1911). The stronger Jordan-Schönflies theorem is false in higher dimension. We will see a proof of it using conformal maps.

10.1. The game of Hex. Consider a finite hexagonal board G: a board whose tiles (or cells) are isometric, regular hexagons. We consider here an $n \times n$ board: there are n hexagons on the North and South side, n on the East and West, and the hexagons on the NW and SE corners have a side in common with two other hexagons, while those on the NE and SW corners have two neighbors. In the game of Hex, the red and blue players take turns putting a stone with their colors on a cell. The red wins if he/she manages to have a path of adjacent red cells joining the North and the South sides, and the blue wins with a blue path joining East and West.

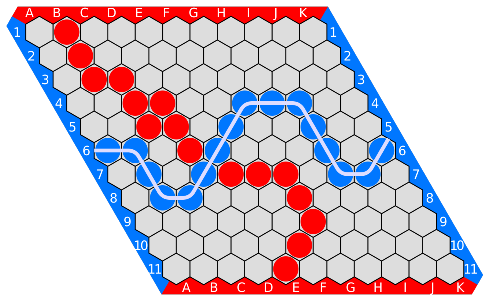

FIGURE 1. By Mliu92 - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=118446451

THEOREM 69 (Hex theorem). *The game of Hex a winner: after a stone has been put on each cell, either there is a winning path for the blue, or there is one for the red.*

The proof just requires all cells to be colored, and not that the number of the red stones equals that of the blue stones. Also, the proof works on any simply connected board the boundary of which is divided into four arcs, meeting at four points, each being interior to some boundary side of a cell. It is intuitive that there can not be two winners, but the proof is more intricate and we do not need it.

PROOF. As in the picture, add four regions $\{E, N, W, S\}$ on the boundary (which we think of as exceptional cells), colored according to the players' goal, in such a way that the NE corner has part of the boundary in common with N and part of it with E, etcetera. Consider the following graph.

- The edges are the sides of the hexagons, plus four exceptional edges separating couples of neighboring boundary regions: $\{E, N\} = EN$, NW, WS, and SE. In the picture, one endpoint of EN and SW is the midpoint of the corresponding corner hexagon's side, but . We consider these two hexagons as if they were pentagons in the obvious way.
- The vertices the vertices of the hexagons, plus a second, exceptional endpoint for each exceptional edge. With some abuse, we use the same symbol NE for the exceptional edge, and for the endpoint on NE lying on the boundary of the board (same for NW, SE, SW).

But for the four exceptional vertices, each vertex is the endpoint of three edges, which separate three regions. We now construct path Γ, which we think of as a sequence $(x_0, x_1, \ldots, x_N)$ of vertices, where x_{i-1} and x_i are endpoints of an edge. We might think of $e_i = (x_{i-1}, x_i)$ as of an oriented edge, and this allows us to talk of the cell *to the right* and of that *to the left* of e_i.

As first edge e_1 is SW, ending in a vertex of the corner hexagon. If the hexagon is red, proceed along the edge separating it from the blue region, and it it is blue, proceed along that separating from the red region Call e_2 this second edge. At the second endpoint of e_2, similarly proceed along the unique edge e_3 which, among the two remaining edges, separates a blue and a red region (there is one and only one, since the two cells meeting along e_2 have different colors). Observe that this way we have a red cell (be it exceptional or not) on the right of each edge of the path, and a blue one on the right.

We claim that the path never crosses the same vertex twice. Suppose it does, and that v is the first vertex which is crossed a second time. Now, v is already the endpoint of two consecutive edges e_i and e_{i+1}, and the third edge is then e_j for some $j > i + 1$ (the first edge hitting the tail). By construction, there are red cells R_i, R_{i+1} on the right of e_i and e_{i+1}, respectively, and blue cells B_i, B_{i+i} on their left. Since there are just three cells meeting at v, either $C_i = C_{i+1}$, or $B_i = B_{i+1}$. In both cases, e_j has cells of the same color on both sides, which is a contradiction.

Since there are finitely many edges and vertices, the only possibility is that Γ ends in one of the three remaining exceptional edges. If the path Γ is only made of boundary edges, it is clear that either red or blue wins. If not, Γ follows for a while (say) the boundary of W, then it enters the board. Let e' be the edge where it enters, and e'' be the first edge where Γ hits again a vertex on the boundary. □

10.2. The Brouwer fixed point theorem in the plane.

THEOREM 70 (Brouwer fixed point theorem in the plane). *Let $f : Q \to Q$ be a continuous map of the unit, closed square $Q = [0, 1] \times [01]$ into itself. Then, there z in Q such that $f(z) = z$.*

By continuity of f and compactness of Q, Brower's theorem follows from its approximate version.

LEMMA 11 (Approximate Brouwer fixed point theorem). *Let $f : Q \to Q$ be a continuous map of the unit, closed square $Q = [0, 1] \times [01]$ into itself. For each $\epsilon > 0$, there is z in Q such that $|f(z) - z| \leq \epsilon$.*

PROOF. □

10.3. The proof of the Jordan curve theorem.

Notes

- Some definitions should be isolated from the main text.
- Add the corners in Green's theorem.

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