Homological and representation stability

Type: PhD course, basic
Sector: MAT/02, MAT/03
Lecturer: L. Caputi
Number of lectures: 16 hours, for a total of 8 meetings
Period: October – December 2024
Exams: seminar form
Preliminaries: Notions of homology groups and basic notions of category theory

Description. The course will introduce the students to the basic theory and results in both homological stability and representation stability. The course will also survey recent developments of the theory and current research directions in the field.

Abstract. Homology is an algebraic invariant that can be defined for several objects: groups, (topological) spaces, algebras, categories. We will be mainly interested in homological invariants of *families* of groups, and in their "stable" homology groups. In fact, homological stability has shown that, for certain families, it is easier to compute infinitely many homology groups at once rather than computing single homology groups proceeding step by step one after the other. Homological stability can be described as follows. Suppose that

$$G_0 \to G_1 \to \ldots \to G_N \to \ldots$$

is a sequence of groups and homomorphism between them. We say that the family $\{G_n\}$ satisfies homological stability if for each q the induced maps $H_q(G_n) \to H_q(G_{n+1})$ are isomorphisms for all $q \leq f(n)$ where $f(n) \to \infty$ with n increasing. Analogously, we can ask the same property for families of topological spaces, algebras, etc. Then, once homological stability for a family is satisfied, one can compute homology groups in the stable range at once. Furthermore, the homology of the limit group can be tackled with homotopy theoretic methods, tools which are inaccessible in the unstable range.

We will deal with families of groups from both algebra and topology, such as the symmetric groups, the braid groups, and the mapping class groups. In fact, homological stability was shown to be satisfied for various families: symmetric groups, braid groups, mapping class groups of surfaces, general linear groups, orthogonal groups, diffeomorphism groups of highly-connected high-dimensional manifolds, to name a few of a big list. Stability techniques in homology of groups started with the intriguing work by Quillen [Qui72] on the computation of K-theory of finite fields; which was also the main motivation. His proof, nowadays known as *Quillen argument* ignited the interest in stability phenomena also in other fields, with many new questions arising in geometric topology.

The main aim of the course is to introduce the fundamental concepts of homological stability, the Quillen argument, to provide the proofs of the main cases, and then to survey similar ideas in related

fields; for example, in representation theory. We now come to the second main theme of the course: Representation stability. Instead of focusing on sequences of groups, we will consider sequences of representations of groups. Beyond the discovery of new phenomena, representation stability can be useful in solving problems outside the theory, eg. in counting problems in number theory. For example, a categorical version of representation stability, as developed by Sam and Snowden, has had unexpected consequences on the general behaviour of the torsion or of the rank growth in homology groups of geometric objects. If time allows, we will conclude the course with showing some recent applications of the theory in the homology of matching complexes and configuration spaces of graphs.

Tentative program.

- Lecture 1 & 2: Homological stability was motivated by problems in K-theory. In these first lectures we first introduce the general concept of homological stability. Then, we focus on the main motivation: K-theory [Qui72, Wei13]. We will introduce the basic concepts and definitions of K-theory and provide some computations.
- Lecture 3 & 4: we start, as warm-up, with the proof of homological stability of braid groups. We outline the Quillen's argument, which is a spectral sequences argument. Therefore, we will first recall some basic notions of spectral sequences needed to carry on the argument, and then provide the proof. If time allows, we will also describe the stable homology of braid groups.
- Lecture 5 & 6: homological stability can be generalized to various automorphism groups of objects living in (well-behaved) categories. In these lectures, we will provide the more general and recent categorical framework due to Randal-Williams and Wahl [RWW17], extending the Quillen argument to objects in homogenous categories. In time allows, we will discuss also the case of twisted coefficients
- Lecture 7 & 8. In these lectures, using the categorical framework, we will focus on homological stability of general linear groups. We will also discuss the case of mapping class groups, and survey recent progresses (eg. in stable homology and functor homology)
- Lecture 9 & 10. In this lecture, we will be interested in representation stability problems [Far14, CF13]. We will describe the difference between homological stability and representation stability, the motivation, and discuss the main examples of configuration spaces and pure braid groups. If time allows, we will discuss the case of Torelli groups.
- Lecture 11 & 12. Representation stability is related to the question whether a certain category of objects is Noetherian; ie, that subfunctors of finitely generated functors are finitely generated. A beautiful recent theory developed by Sam and Snowden [SS17] tackles this problem from a combinatorial point of view. In these lectures we will introduce the notion of Gröbner categories, and discuss the relation between Noetherianity of categories and Gröbner categories.
- Lecture 13 & 14. In these lectures, we will apply the theory developed by Sam and Snowden to the category of graphs. We will see that, when bounding the genus, (representation categories of) categories of graphs are Noetherian. We will apply it to the case of matching complexes.

- Lecture 15 & 16. In these lectures, we will conclude with a survey of more recent developments in homological stability and representation stability.

References

[CF13] Thomas Church and Benson Farb. Representation theory and homological stability. *Advances in Mathematics*, 245:250–314, 2013.

[Far14] Benson Farb. Representation stability. *arXiv preprint arXiv:1404.4065*, 2014.

[Qui72] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. Math. (2)*, 96:552–586, 1972.

[RWW17] Oscar Randal-Williams and Nathalie Wahl. Homological stability for automorphism groups. *Adv. Math.*, 318:534–626, 2017.

[SS17] S. V. Sam and A. Snowden. Gröbner methods for representations of combinatorial categories. *J. Amer. Math. Soc.*, 30(1):159 – 203, 2017.

[Wei13] Charles A. Weibel. *The K-book. An introduction to algebraic K-theory*, volume 145 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2013.