Topological Dynamics and Universal Minimal Flows

Type: PhD course, basic
Sector: MAT/01, MAT/03
Lecturer: A. Codenotti
Number of lectures: 8 lectures, for a total of 16 hours
Period: February-May 2025
Exams: Seminar
Prerequisites: Basic knowledge of topological groups and general topology

Description: The course will introduce the students to the basic notion of abstract topological dynamics, with a focus on universal minimal flows. Apart from a few foundational results the theory is relatively recent, having been developed mostly in the last 25 years, so the course will focus on recent developements and current research trends.

Abstract: Given a topological group G, a *G-flow*, or simply a *flow* when the group is clear from context, is a continuous action $G \curvearrowright X$ on a compact Hausdorff space. A flow is called *minimal* if all the orbits are dense, or equivalently if there is no proper closed subspace of X which is invariant under the action of G. Given two G-flows X and Y, a *morphism of G-flows* is a continuous map $f\colon X \to Y$ which is G-equivariant, meaning that

$$f(g \cdot x) = g \cdot f(x),$$

for every $g \in G$ and every $x \in X$. If there is a surjective morphism $f\colon X \to Y$ we say that Y is a *factor* of X. By a classical result of Ellis every topological group G has a *universal minimal flow*, that is a minimal G-flow which has all the other minimal G-flows as factors, moreover this flow is unique up to isomorphism [Ell60, Theorem 2], so that we can talk about *the* universal minimal flow of G, denoted by $M(G)$.
This is a central object of study in abstract topological dynamics, with much effort devoted to explicitly describing $M(G)$ for a group G of interest, which is often, but not always, a *Polish group*, that is a completely metrizable and separable group. This goal is often not attainable, since many groups G have a universal minimal flow which is not metrizable and is instead a huge space containing a copy of $\beta\mathbb{N}$, the Stone-Čech compactification of the integers, making

an explicit description impossible. As we will see during the course this is the case for all infinite discrete groups and more generally for all locally compact noncompact groups [KPT05, Theorem A2.2] but, surprisingly, there are groups G for which $M(G)$ is not only metrizable, but the trivial space with one point. Those groups are called *extremely amenable* and can be equivalently characterized as the groups enjoying the following fixed point property: G is extremely amenable if and only if every G-flow has a fixed point. There has been a growing interest in recent years in finding groups that have a universal minimal flow which can be explicitely understood, in particular a lot of research has been devoted to establishing which Polish groups have a metrizable universal minimal flow.

The first goal of the course will be to introduce the basic notions of topological dynamics, to prove all the results mentioned above, and to give an explicit example of an extremely amenable group. This will be $\mathrm{Aut}(\mathbb{Q}, \leq)$, the automorphism group of the rationals as a linear order, equipped with the (Polish) topology of pointwise convergence, whose extremely amenability was first establishes by Pestov [Pes98, Corollary 5.6]. As a corollary of Pestov's result we will obtained a characterization of $M(\mathrm{Homeo}^+(S^1))$, the universal minimal flow of the group of orientation-preserving homeomorphism of S^1, equipped with the compact-open topology. More explicitely we will show that $M(\mathrm{Homeo}^+(S^1))$ can be identified with the natural action $\mathrm{Homeo}^+(S^1) \curvearrowright S^1$, another result due to Pestov. This leads us to the second part of the course, in which we will focus on the case of manifolds. Inspired by the result about S^1 mentioned above Pestov asked whether $M(\mathrm{Homeo}(X))$ can be identified with the natural action $\mathrm{Homeo}(X) \curvearrowright X$ when X is a closed orientable manifold of dimension at least 2. This was answered negatively by Uspenskij [Usp00], by introducing what has now become a central object in the study of universal minimal flows: the space of chains over M. We will give a proof of Uspenskij's result and, to conclude the course, we will work toward a strenghtening of it by Gutman, Tsankov and Zucker [GTZ21], who showed that $M(\mathrm{Homeo}(X))$ is not metrizable whenever X is a compact manifold of dimension at least 3. If time permits we will sketch a further strenghtening of this result by Basso, Codenotti and Vaccaro [BCV24].

Tentative Program:

- Lecture 1: In this lecture we will introduce the basic notions of topological dynamics and prove existence and uniqueness of universal minimal flows. Some examples of universal minimal flows (without proofs) will be given.
- Lecture 2: We start with a negative result: locally compact noncompact groups are never extremely amenable. If time permits we will strenghten this result to locally compact noncompact groups never have a metrizable universal minimal flow.
- Lecture 3: In this lecture we will prove Pestov's result that $\mathrm{Aut}(\mathbb{Q}, \leq)$ is extremely amenable, and obtain a characterization of $M(\mathrm{Homeo}^+(S^1))$ as a corollary.

- Lecture 4: We begin this lecture by introducing the Vietoris topology, the space of chains and describing their most important properties. We will then prove Uspenskij's result that $M(\text{Homeo}(X))$ is not the natural action $\text{Homeo}(X) \curvearrowright X$ when X is a compact manifold of dimension at least 3.

- Lecture 5: This lecture will be devoted to a few lemmas of indepent interest that will be needed in the remainder of the course. We will see a criterion due to Rosendal to check that an action has meager orbits. We will then leverage Rosendal's result together with some results by various authors (Ben Yaacov, Mellerey, Tsankov, Zucker) to obtain a concrete strategy which can be used to show that $M(G)$ is not metrizable when G is a Polish group.

- Lectures 5 to 8: The rest of the course will focus on proving a recent result by Gutman, Tsankov and Zucker [GTZ21], who showed that $M(\text{Homeo}(X))$ is not metrizable whenever X is a compact manifold of dimension at least 3. If there is enough time we will also sketch a further strenghtening of this result by Basso, codenotti and Vaccaro [BCV24], in which the same nonmetrizability result is established for groups of the form $\text{Homeo}(X)$ for a much wider class of spaces X.

References

[BCV24] Gianluca Basso, Alessandro Codenotti, and Andrea Vaccaro. *Surfaces and other Peano Continua with no Generic Chains*. 2024. arXiv: `2403.08667 [math.DS]`. URL: `https://arxiv.org/abs/2403.08667`.

[Ell60] Robert Ellis. "Universal minimal sets". In: *Proceedings of the American Mathematical Society* 11.4 (1960), pp. 540–543.

[GTZ21] Yonatan Gutman, Todor Tsankov, and Andy Zucker. "Universal minimal flows of homeomorphism groups of high-dimensional manifolds are not metrizable". In: *Math. Ann.* 379.3-4 (2021), pp. 1605–1622. ISSN: 0025-5831. DOI: `10.1007/s00208-021-02146-1`. URL: `https://doi.org/10.1007/s00208-021-02146-1`.

[KPT05] Alexander S. Kechris, Vladimir G. Pestov, and Stevo Todorčević. "Fraïssé Limits, Ramsey Theory, and topological dynamics of automorphism groups". In: *Geometric & Functional Analysis GAFA* 15 (2005), pp. 106–189.

[Pes98] Vladimir G. Pestov. "On free actions, minimal flows, and a problem by Ellis". In: *Trans. Amer. Math. Soc.* 350.10 (1998), pp. 4149–4165. ISSN: 0002-9947. DOI: `10.1090/S0002-9947-98-02329-0`. URL: `https://doi.org/10.1090/S0002-9947-98-02329-0`.

[Usp00] Vladimir Uspenskij. "On universal minimal compact G-spaces". In: *Proceedings of the 2000 Topology and Dynamics Conference (San Antonio, TX)* 25 (2000), pp. 301–308.