SINGULAR INTEGRAL OPERATORS IN HARMONIC ANALYSIS

JOAN VERDERA

Abstract of the course. This course aims to introduce some fundamental topics in modern harmonic analysis and to emphasise the import of these topics on the partial differential equation applications, in particular on the study of free boundary value problems. The course will be taught by a renown specialist: Joan Verdera (U. Autonoma Barcelona); the course is designed to be the natural continuation of the elective MS course on *Real & Harmonic Analysis* taught by Loredana Lanzani, which is attended on average by 30 students/year.

Period: from April to May 2025 (six weeks)

This 6 weeks-long course will focus on a selection of topics from the Calderón-Zygmund Theory of Singular Integral operators and applications to the solution of boundary value problems for harmonic functions on Lipschitz domains: a fundamental part of modern analysis and the gateway to the contemporary developments on PDE over uniformly rectifiable domains. The course will start with the so-called first generation Calderón-Zygmund (CZ) operators, namely convolution operators with homogeneous kernels (which are smooth off the origin and have vanishing integral on the unit sphere). The L^2 theory is presented via the computation of the symbol and leads to the heart of the matter: the CZ decomposition and its consequences, namely, the L^1 - to - weak L^1 estimate and, via interpolation, the L^p boundedness, $1 < p < \infty$. At the other extreme we discuss the L^∞ - to- BMO estimate. The maximal singular integral then leads to pointwise convergence of truncations via Cotlar's inequality. We will then make a brief excursion in the non translation invariant theory for the Cauchy integral on Lipschitz graphs and Calderón commutators, where one looses, a priori, the Fourier transform and novel techniques must be developed. We will then treat Sobolev spaces with emphasis on the Hardy-Littlewood-Sobolev immersion theorem and then follow with a brief treatment of differentiabilty properties of Sobolev functions, a topic close to Zygmund's and Stein's interests. Along the way we discuss several concrete examples of CZ operators that set the ground for deep applications to PDE: the boundary layer potential technique for solving the Dirichlet and Neumann boundary value problems for harmonic functions on Lipschitz domains.

Didactic information

Proposed Syllabus

(1) Review of the Hilbert and Beurling transforms as the "primordial" singular integral operators: the role of the Fourier transform; connection with complex analysis.
(2) First generation CZ operators: L^2-theory; L^1 - to - weak L^1; interpolation. BMO estimates.
(3) The maximal singular integral and Cotlar's inequality
(4) The Cauchy Transform on Lipschitz graphs

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(5) Notable examples: $\overline{\partial}$-operator; the second order Riesz transforms and their relation to the Laplacian and Sobolev spaces; the single-, double- , and boundary-layer potential operators associated to a Lipschitz domain in Euclidean space.
(6) Applications to PDE: solvability of the L^p-Dirichlet and L^p-Neumann boundary value problems for harmonic functions on Lipschitz domains. Representation of the solutions via the aforementioned layer potential operators. Optimality of the $p-$range and operator bounds within the Lipschitz category.

Didactic assessment. the teacher will assign course grades accordingly with a procedure that will be communicated before the beginning of the course.

References

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[8] P. Mattila, M. S. Melnikov & J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability, Annals of Mathematics. Second Series, 144, (1996) 127-136.
[9] Verchota G., Layer Potentials and regularity for the Dirichlet problem for Laplace's equation on Lipschitz domains, J. Functional Analysis, 59, (1984) 572 - 611.

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