UNIFORM AND COARSE STRUCTURES ON POLISH GROUPS

Type: PhD course, basic
Sector: MAT/01, MAT/02, MAT/03
Lecturer: J. Zielinski
Number of lectures: 16 hours, for a total of 8 meetings
Period: February – June 2025
Assessment: Graded problems
Preliminaries: Enough elementary general topology and group theory to comprehend the basic definitions and constructions. A background in logic is not a prerequisite, although there will be places—especially in part II—where some acquaintance with basic model theory would lend familiarity to the techniques and help motivate the significance of certain results.

Abstract. There is a canonical quartet of uniform structures associated to every topological group, derived by applying the group operations to sets in a neighborhood basis of the identity element. Therefore, one can speak not only of group-theoretic and topological properties of such a group, but also unambiguously of uniform notions such as Cauchy sequences, uniform continuity, and precompactness (provided that one specifies which of the four uniformities, is being considered e.g., "left Cauchy sequences" or "Roelcke precompact sets").

Likewise, given an appropriately compatible ideal of bounded sets, the multiplication operation automatically upgrades this structure to a collection of four compatible coarse structures, allowing one to speak of stronger notions like bornologous maps or spaces having bounded geometry for any group equipped with such an ideal.

This class will be grouped into three parts. In the first and shortest part, we will review the basic notions of topological groups, with an emphasis on Polish groups and non-archimedean Polish groups. In the second part we will introduce the four canonical uniform structures on such groups, with an emphasis on the left and Roelcke uniform structures. Every Polish (resp. non-archimedean Polish) group admits a continuous and topologically faithful isometric action on a separable, complete metric space (resp. a countable set), and we will see examples of how uniform properties of such groups connect to properties of these actions. In the third and longest part we will discuss the ideal of *coarsely bounded sets* for a topological group identified by C. Rosendal, and explore his theory of the left coarse structure associated to this ideal.

Tentative program.

Part I: Topological groups. Lectures # 1 – # 2

Here we will recall fundamental notions of topological groups and of continuous group actions and define the subclasses of Polish groups and non-archimedean Polish groups. We will illustrate how the non-archimedean groups can be realized as the automorphism groups of countable structures and Polish groups generally as the automorphism groups of continuous (metric) structures. We may also discuss some Baire category techniques, the existence of universal Polish groups, and other fundamental results about Polish groups that will be useful in the later parts of the course.

Part II: Uniform structures on topological groups. Lectures # 2 – # 4

Next we will introduce the four canonical uniform structures on a topological group and associated properties, such as being (Weil) complete, Roelcke precompact, or SIN (having Small Invariant Neighborhoods). Here the main reference is [RD81]. We will emphasize the connection, for $\mathcal{M}$ a countable discrete or separable metric structure, between uniform properties of $\mathrm{Aut}(\mathcal{M})$ and the fundamental action of $\mathrm{Aut}(\mathcal{M}) \curvearrowright \mathcal{M}$. For example, S. Gao's observation [Gao98] that a countable structure $\mathcal{M}$ admits a non-surjective $\mathcal{L}_{\omega_1,\omega}$-elementary self embedding if and only if $\mathrm{Aut}(\mathcal{M})$ is not complete in the left uniformity, or the continuous version of the Engeler–Ryll-Nardzewski–Svenonius characterization of separably-categorical structures from [BYT16, Ros13, Tsa12] or, if time permits, the characterization of T. Tsankov and I. Ben Yaacov [BYT16] relating the stability of such categorical structures $\mathcal{M}$ to the algebra of weakly almost periodic functions on $\mathrm{Aut}(\mathcal{M})$.

Part III: Coarse geometry of topological groups. Lectures # 5 – # 8

Finally, we will consider the coarse geometry of Polish groups. This still is a relatively new theory with myriad potential applications such as, in recent years, to the study of "big" mapping class groups. Here the primary reference will be [Ros21] and the outline of topics is:

- Definitions and examples of the coarsely bounded subsets of a topological group, the left coarse structure, coarse maps, coarse equivalences, locally bounded groups (i.e., groups with metrizable coarse geometry), groups generated by a coarsely bounded set (i.e. groups with a well-defined quasi-isometry type).
- Locally Roelcke precompact groups and the characterization of these groups in terms of their uniform completions and their coarse geometry (from [Zie21])
- Bounded geometry for topological groups, the characterization of Polish groups of bounded geometry as those admitting a continuous, coarsely proper, modest, cocompact action on a locally compact metric space, and the theorem that coarse equivalence for such groups is tantamount to the existence of commuting continuous, coarsely proper, modest, cocompact actions on a locally compact Hausdorff space (all from [Ros21]).

References

[BYT16] Itaï Ben Yaacov and Todor Tsankov. Weakly almost periodic functions, model-theoretic stability, and minimality of topological groups. *Transactions of the American Mathematical Society*, 368(11):8267–8294, 2016.

[Gao98] Su Gao. On automorphism groups of countable structures. *The Journal of Symbolic Logic*, 63(3):891–896, 1998.

[RD81] Walter Roelcke and Susanne Dierolf. *Uniform structures on topological groups and their quotients*. McGraw-Hill College, 1981.

[Ros13] Christian Rosendal. Global and local boundedness of Polish groups. *Indiana University Mathematics Journal*, pages 1621–1678, 2013.

[Ros21] Christian Rosendal. *Coarse geometry of topological groups*, volume 223. Cambridge University Press, 2021.

[Tsa12] Todor Tsankov. Unitary representations of oligomorphic groups. *Geometric and Functional Analysis*, 22(2):528–555, 2012.

[Zie21] Joseph Zielinski. Locally Roelcke precompact Polish groups. *Groups, Geometry, and Dynamics*, 15(4):1175–1196, 2021.