Title: Simplicial Homotopy Theory
Type: PhD course, advanced
Sector: MAT/03
Lecturer: Nicholas Meadows
Referent: Martino Lupini
Number of Lectures: 8 lectures for a total of 16 hours
Period: I semester
Exams: Seminar
Prerequisites: familiarity with basic algebraic topology.

Description This course will provide an introduction to the homotopy theory of simplicial sets, which both provide combinatorial models for spaces up to homotopy (e.g. Kan complexes), as well models of (∞, 1)-categories (through Joyal/Lurie's quasi-categories). We will also provide applications to equivariant homotopy theory and K-theory.

Abstract Since the 1970s, simplicial homotopy theory has played a critical role in the foundations of algebraic topology and algebraic geometry. Early important examples of the applications of simplicial methods include Bousfield and Kan's work on the theory localizations of topological spaces ([BK72]), and Quillen's work on the foundations of algebraic K-theory ([Qui73]). More recent examples include the motivic homotopy of Voevodsky ([MV99]), and the work of Lurie on derived algebraic geometry in [Lurb]. In support of his work on derived algebraic geometry, Lurie wrote thousands of pages on the foundations of the theory of ∞-categories, primarily in [Lura] and [Lur09].
The main purpose of this course will be to give a basic foundation in simplicial homotopy so that students will have the necessary background to understand applications of simplicial method in their own field. However, we will provide examples related to K-theory and equivariant homotopy theory to illustrate the use of the theory.
The first portion of the course will be devoted to explaining how the homotopy theory of simplicial sets subsumes model the classical homotopy theory of topological spaces. We will explain how important constructions, such as homotopy groups, fibrations, Eilenberg-Maclane spaces, etc. can be constructed in the simplicial setting.
In the second part of the course, we will focus on homotopy colimits and their applications. In particular, we will describe how Quillen's Theorem A and B apply to the foundations of unstable equivariant homotopy theory and algebraic K-theory.
In the final part of the course, we will explain how ∞-categories can as models of 'homotopy theories.' The examples we will focus on will be the standard and equivariant stable homotopy categories. The latter has had renewed interest due to the importance of equivariant homotopy in the resolution of important topological problems in recent years, such as the Kervaire invariant one problem [HHR09].

Tentative Lecture plan

- Lecture 1: In this lecture, we will describe basic notions of simplicial sets, such as Kan fibrations, function complexes, simplicial homotopy groups, and geometric realizations
- Lecture 2: We will describe classical constructions from topology in the context of simplicial sets, such as Postnikov towers and the Hurewicz homomorphism. We will also briefly describe the model structure on simplicial sets.

- Lecture 3-4: This lecture will cover homotopy colimits, Quillen's theorem A and B, and classical applications to algebraic K-theory. The latter includes the group completion theorem, Waldhausen K-theory, Devissage.
- Lecture 5: Applications to the foundations of equivariant homotopy theory. Elmendorf's theorem and Bredon cohomology. Sample calculations.
- Lecture 6. An introduction to quasi-categories. Basic constructions such as slice and join and limits.
- Lecture 7: The ∞-categories of spaces and spectra. Construction of the equivariant stable homotopy category.
- Lecture 8: Foundations of equivariant stable homotopy theory via∞-categories (Burnside categories, mackey functors, etc.).

References

[BG16] Clark Barwick and Saul Glasman. On the fibrewise effective Burnside ∞-category. 2016.

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[GJ09] Paul G. Goerss and John F. Jardine. Simplicial homotopy theory. Modern Birkh¨auser Classics. Birkh¨auser Verlag, Basel, 2009.

[HHR09] M.A. Hill, M. J. Hopkins, and D.C Ravenal. on the nonexistence of elements of kervaire invariant one. Annals of Mathematics, 184(1):1–262, January 2009.

[Lura] Jacob Lurie. Higher Algebra.

[Lurb] Jacob Lurie. Spectral Algebraic Geometry.

[Lur09] Jacob Lurie. Higher topos theory, volume 170 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.

[May96] J. P. May. Equivariant homotopy and cohomology theory, volume 91 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996.

[MV99] F. Morel and V Voevodsky. A1-homotopy theory of schemes. Publications Math´ematiques de l'IH ´ ES,, 90:45–143, 1999.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[Wei13] Charles A. Weibel. The K-book: An Introduction to Algebraic K-theory. American Mathematical Soc., June 2013.