

AN INTRODUCTION TO FREE BOUNDARY PROBLEMS

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This is an introductory course in the theory of free boundary problems, with emphasis on regularity questions. We plan to discuss two relevant classical problems, the Obstacle Problem and the Bernoulli Problem.

The study of the obstacle problem originated in the context of elasticity as the equations that models the shape of an elastic membrane which is pushed by an obstacle from one side affecting its shape. The resulting equation for the function whose graph represents the shape of the membrane involves two distinctive regions. In the part of the domain where the membrane does not touch the obstacle, the function will satisfy an elliptic PDE. In the part of the domain where the function touches the obstacle, the function will be a supersolution of the elliptic PDE. Everywhere, the function is constrained to stay larger or equal than the value of the obstacle. More precisely, there is an elliptic operator L and a function φ (the obstacle) so that

$$Lu \leq 0, \quad u \geq \varphi \quad \text{in } D$$

and

$$Lu = 0 \quad \text{in } \{u > \varphi\} \cap D.$$

In the simplest model, $L = \Delta$.

Similarly, Bernoulli's free-boundary problem arises in ideal fluid dynamics, optimal design, electro-chemistry, electro-statics, and further applications. In its simplest form, it consists of finding a solution to:

$$Lu = 0 \quad \text{in } \{u > 0\} \cap D$$

and

$$|\nabla u|^2 = g(x) \quad \text{on } F(u) := \partial\{u > 0\} \cap D,$$

for a given $g > 0$.

We will consider the model case $L = \Delta$ and discuss the following issues for each problem:

- (i) Existence of solutions via variational techniques.
- (ii) Optimal regularity of the solution.
- (iii) Regularity of the free boundary.
- (iv) Structure of the singular set.

To cover these topics, classical results of independent interest will be discussed, such as Boundary Harnack Inequality, Perturbation Theory, Monotonicity Formulae.

A few standard references are listed below, however many more resources will be used.

REFERENCES

- [1] H.W. Alt; L.A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*. Journal für die reine und angewandte Mathematik (1981) Volume: 325, page 105-144.
- [2] L. A. Caffarelli, *The obstacle problem revisited*, Journal of Fourier Analysis and Applications volume 4, pages 383–402(1998)
- [3] L.A. Caffarelli, S. Salsa, *A Geometric Approach to Free Boundary Problems*, Graduate Studies in Mathematics, vol. 68.
- [4] A. Petrosyan, H. Shahgholian, N. Uraltseva, *Regularity of Free Boundaries in Obstacle-Type Problems*. Graduate Studies in Mathematics Volume: 136; 2012; 221pp.